F.E.D. Vignette #18 --

The Dialectic of 'The Basic Operations of Elementary Arithmetic' --

Systematically Presented via a 5-Symbol Expression.

by Miguel Detonacciones

Author's Preface. The purpose of **F.E.D.** Vignette **#18** is to provide another, simple, "worked example" of a ***systematic-dialectical equation-model*** and of the general **E.D.** method of solution of ***dialectical*** ['***meta-***]***models***'.

A Note about the On-Line Availability of Definitions of F.E.D. Key Technical Terms. Definitions of ***Encyclopedia Dialectica*** technical terms, including of **E.D.** 'neologia', are available on-line via the following URLs --

http://www.dialectics.org/dialectics/Glossary.html

https://www.point-of-departure.org/Point-Of-Departure/ClarificationsArchive/ClarificationsArchive.htm

-- by clicking on the links associated with each such term, listed, in alphabetic order, on the web-pages linked-to above.

Links to definitions of the ***Encyclopedia Dialectica*** special terms most fundamental to this vignette are as follows --

«***aufheben***»
https://www.point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Aufheben/Aufheben.htm

Diachronic vs. ***Synchronic***
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Synchronic/Synchronic.htm

Dyadic Seldon Function as '''Self-Reflexive Function''' and as 'Self-Iteration'
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/SeldonFunctions/SeldonFunctions.htm
http://www.dialectics.org/dialectics/Glossary_files/%27Dyadic%20Seldon%20Functions%27_as_%27Self-Iterations%27,_vs._Standard_%27Other-Iterations%27.jpg

«***genos***»
https://www.point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Genos/Genos.htm

Historical or ***Diachronic Dialectics***
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/HistoricalDialectics/HistoricalDialectics.htm

$_{\mathbf{N}}\mathbf{Q}$ ***dialectical arithmetic / algebra***
http://www.dialectics.org/dialectics/Correspondence_files/Letter17-06JUN2009.pdf

«***species***»
https://www.point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Species/Species.htm

Systematic or ***Synchronic Dialectics***
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/SystematicDialectics/SystematicDialectics.htm

-- and we plan to expand these definitions resources as the ***Encyclopedia Dialectica Dictionary Project*** unfolds.

[**Note**: '''Arithmetical Quantifiers''' vs. 'Arithmetical Qualifiers'. In the phrase "**3** apples", we term "**3**" the "arithmetical ["pure"-]***quant***ifier", and "apples" the '''***ontological***''' -- or ***kind*** of thing -- '''***qual***ifier'''. In the phrase "**3** pounds of apples", we term "pounds" the '***metrical***[-unit] ***qual***ifier' -- or '''unit of measure ***qual***ifier''' -- ***quant***ified by the **3**, which, together, '***quanto-qualify***' the '***ontological qual***ifier', "apples". A key use-value of the ***dialectical arithmetics*** is to provide algorithmic, ideographical-symbolic systems for the various kinds of 'arithmetical ***qual***ifiers', both with and without the co-presence of '''arithmetical ***quant***ifiers'''.].

Introduction. This model is more "**C**omplex" [pun intended] than the models of "TV-Series", and of 'Modern Computerware', presented earlier in this sub-series, because it requires some "domain-expertise" -- or, at least, some "domain familiarity" -- with respect to the domain of the so-called "**C**omplex Numbers", the set standardly denoted by the symbol **C**. The 'axioms-system' of the arithmetic of the **C** numbers, which we denote by **C**, is the **6**th system of arithmetic in the following standard order of standard arithmetics, with our light-spectrum **ordinal** color-coding added: **N**, **W**, **Z**, **Q**, **R**, **C**, for the "**N**atural", "**W**hole", "Integer", "Rational", "**R**eal", and "**C**omplex" arithmetics, respectively. About the 'Goedelian Dialectic' of these systems, see: http://www.dialectics.org/dialectics/Vignettes.html, Vignette #**4**.

We will, in this vignette, use the **C**omplex-Numbers-subsuming version of the **F**.*E*.*D*. '***first dialectical algebra***' to construct, and to "solve", a "heuristic", 'intuitional' model of a ***systematic presentation*** of the domain of "the basic **O**perations of arithmetic" -- encompassing both its '''*verse*''' [e.g., addition, multiplication, exponentiation] and its *"inverse"* [e.g., subtraction, division, root-extraction] operations, jointly, via **q**s with **C** subscripts, which we also reference as $_{c}$**q**s.

The models that we usually narrate here are constructed by interpreting the ***gene***ric $_{N}$**Q** version of the **F**.*E*.*D*. '***first dialectical algebra***' [see E.D. Brief # 5 and its Preface], or, at most-advanced, by interpreting the ***gene***ric $_{W}$**Q** version of that algebra [see E.D. Brief #6 and its Preface], with the subscripts of the $_{N}$**q** or $_{W}$**q** 'meta-numerals' drawn, therefore, from the number-space **N** ≡ { **1**, **2**, **3**, **...** }, or from the number-space **W** ≡ { **0**, **1**, **2**, **3**, **...** }, respectively.

This time, the subscripts of the $_{c}$**q**s will be drawn from the standard number-space **C** ≡ { **R** + **Ri** }, wherein **R** denotes the space of the standard so-called "**R**eal" numbers, and where the **i** unit stands for so-called "**i**maginary" unity, the positive square root of **–1**. FYI: The ***gene***ric **C**omplex number is often expressed as **a** + **bi**, with **a** an element of the set **R**, and also with **b** an element of the set **R**, or as **z** = **x** + **yi**, with **x** an element of **R**, and also with **y** an element of **R**.

That is, we will be constructing our example model using the ***gene***ric $_{c}$**Q** version of the **F**.*E*.*D*. '***first dialectical algebra***'.

We use the $_{c}$**Q** language this time, as it allows us to present both '''*verse*''' & '''*reverse*''' operations in a single model.

Herein we mean, by the word, '''***systematic***''' in the phrase '''***systematic*** presentation''', a presentation of the major kinds of "entities" that exist in this domain, the domain of the basic operations of arithmetic -- by means of categories that classify those entities by their meme *"kinds"*, i.e., as '''[ideo-]*ontology*''', or as *"kinds* of [idea-]things" -- and in strict order of rising complexity / inclusiveness, starting from the simplest category, and moving, step-by-step, from lesser to greater complexity, i.e., more inclusive complexity, until we reach the most complex / inclusive extant category of this domain, or for the purposes of this example.

The model that we build will describe these categories in that strict, systematic order of rising operational complexity.

This model will be, once again, like the previous two "simpler" models, a "snapshot" model, a "synchronic" model that takes the contemporary slice of time -- or, at any rate, a recent-past slice of time -- and algorithmically ***gene***rates descriptions of categories for entities that presently exist, or that might possibly presently exist, for the model's domain, in their systematic order of inclusivity / complexity, as described above.

Our model here will ***not*** be a "chronology" model, or "diachronic" model, like our model of the ***psychohistorical dialectic of human-social formation***, in which the units of *earlier* categories are described as actually, e.g., physically, ***constructing***, through their ***activity*** as *"**causal agents**"*, i.e., as *"**subjects**"*, the units of *later* categories, categories whose units ***did not exist*** until that construction took place. That is, it will ***not*** be a model of a 'self-advancing' ***historical progression*** of ontology, with each historical epoch containing both old ontology, inherited from past historical epochs, plus new ontology, ontology that had ***never appeared before*** -- in past historical epochs -- ***until*** the *later* epoch in question, plus also 'hybrid categories', combining / synthesizing the old ontology/categories with the new.

We will apply a documented, standard procedure to "solve" this '''algebraic''' model -- to determine what actual category each of these generated category-descriptions refers to, and to determine which, if any, of these category-descriptions describe "empty categories", i.e., represent 'combinatorially' *possible* entities that *actually* do not exist for this domain -- at least not presently.

To get started, we must determine the starting-point -- the point-of-departure -- for our ***systematic model***.

This ***starting category*** will be the seed of our whole progression of generated category-descriptions, influencing every category that follows, as the "controlling ***source***", and as the "ever-present ***origin***", of all that follows from it.

The rule for getting started is to ask oneself "¿What is the ***least complex*** kind of thing, the ***simplest*** kind of thing, the least inclusive kind of thing, which inheres in this domain***?***" -- in our case, in the domain of 'basic arithmetic operations' -- and to then find the answer to that question, based upon one's prior knowledge of, or familiarity with, this domain.

The answer to this starting question that we will pursue in this example is the following: The '''**verse**''' operations of "**Additions**", and its "***in***verse operations", or '''***re***verse operations''', of "**Subtractions**", are the simplest ancestors, the ultimate units, of basic arithmetical operations, ingredient in every one of the more complex operations of that domain.

A letter that the spelled names of these two kinds of operations have in common is "**t**".

Therefore, we shall name/symbolize our ***starter category*** as $_{c}\mathbf{t}$, or as $_{c}\mathbf{q}_{\mathbf{t}}$, denoting the "**C**omplex" combination of the "**Additions**" sub-category of elementary **R**eal arithmetic basic operations, with the sub-category of "**Subtractions**", and identifying that combination of ***specific*** sub-categories with the ***generic first*** category symbol of our ***generic*** category-arithmetic model, namely, with the symbol $_{c}\mathbf{q}_{[1 + 1i]}$, in an "identification", an "interpretation", or an "assignment" [all denoted by '**[---)**'] that we indicate by writing: $_{c}\mathbf{t} \equiv {_{c}\mathbf{q}_{\mathbf{t}}} \equiv {_{c}\mathbf{q}_{[A + Si]}}$ **[---)** $_{c}\mathbf{q}_{[1 + i]}$.

Our model, then, will take the form of an "interpreted", ***specific*** equation, assigned to the ***generic*** equation, like this --

$$_{c})\text{-}|\text{-}(_{s} = {_{c}\mathbf{t}}^{2^{s}} \equiv ({_{c}\mathbf{q}_{[A + Si]}})^{2^{s}} \quad [---) \quad {_{c}|\text{-}|\text{-}|_{h}} = {_{c}\mathbf{h}}^{2^{h}} \equiv [{_{c}\mathbf{q}_{[1 + 1i]}}]^{2^{h}}$$

-- with the variable **s** indicating the **s**tep in our ***systematic method of presentation*** that the 'accumulation of categories', denoted by $_{c})\text{-}|\text{-}(_{s}$, represents. We will not, here, further recount the [Marxian] method of ***systematic discovery*** that was used to arrive at the ***starting category*** of this ***systematic presentation***. For more regarding this ***method of discovery***, see Marx, ***Grundrisse***, Penguin Books [London: **1972**], pp. **100-101**.

Stage 0. Our initial **s**tep -- **s**tep **s = 0** -- contains only our ***starting category***, $_{c}\mathbf{t} \equiv {_{c}\mathbf{q}_{\mathbf{t}}} \equiv \mathbf{q}_{[A + Si]}$ --

$$_{c})\text{-}|\text{-}(_{0} = {_{c}\mathbf{t}}^{2^{0}} = {_{c}\mathbf{t}}^{1} = {_{c}\mathbf{t}} \equiv \mathbf{q}_{[A + Si]} \quad [---) \quad {_{c}\mathbf{q}_{[1 + 1i]}}$$

-- because **2** "raised" to the power **0** -- $\mathbf{2^0}$ -- is just **1**, and because $_{c}\mathbf{t}$ "raised" to the power **1** is just $_{c}\mathbf{t}$.

Stage 1. It is when we get to the next step after step $s = 0$, namely, to step $s = 1$, that our equation-model gives us back something initially "unknown" -- and, therefore, something '''algebraical''', not merely something '''arithmetical''': something to "solve-for" --

$$)\text{-}|\text{-}(_1 \;=\; {}_c\underline{t}^{2^1} \;=\; {}_c\underline{t}^{2} \;=\; {}_c\underline{t} \times {}_c\underline{t} \;=\; {}_c\underline{q}_{[A + Si]} \times {}_c\underline{q}_{[A + Si]} \;\equiv$$

$${}_c\underline{q}_{[A + Si]} + {}_c\underline{q}_{[AA + SSi]} \;=\; {}_c\underline{q}_{t} + {}_c\underline{q}_{tt}$$

-- because 2 "raised" to the power 1 -- 2^1 -- is just 2, and because our rule for multiplying a generic category, call it ${}_c\underline{q}_{[X + Yi]} \equiv {}_c\underline{q}_{Z} \equiv {}_c\underline{Z}$, "by", or "into", itself, is, for subscripts X and Y denoting *sub*-category symbols, and for subscript Z denoting a *category*-symbol, simply --

$${}_c\underline{q}_{[X + Yi]} \times {}_c\underline{q}_{[X + Yi]} \;\equiv\; {}_c\underline{q}_{[X + Yi]} + {}_c\underline{q}_{[XX + YYi]} \;\equiv\; {}_c\underline{Z} + {}_c\underline{q}_{ZZ}$$

-- and for x and y denoting "**R**eal" numbers --

$${}_c\underline{q}_{[x + yi]} \times {}_c\underline{q}_{[x + yi]} \;\equiv\; {}_c\underline{q}_{[x + yi]} + {}_c\underline{q}_{[(x + x) + (y + y)i]} \;=\; {}_c\underline{q}_{[1x + 1yi]} + {}_c\underline{q}_{[2x + 2yi]}.$$

Note again: Herein, the ${}_c\underline{q}$ symbol-component denotes the ***generic*** category '**q**ualifier' with "**C**omplex" subscripts.

The subscripts that come after it are ***specific*** category descriptors.

¿But how do we discover what the resulting, added, "unknown"/'algebraical' 'category-description', ${}_c\underline{q}_{tt}$, ***means?***

Well, the ***generic*** rule to "solve-for" the categorial ***meaning*** of such symbols is that, if we know what is meant by category ${}_c\underline{q}_{z} = {}_c\underline{z}$, then the symbol ${}_c\underline{q}_{zz}$ describes a category each of whose units is a '${}_c z$ ***OF*** ${}_c z$s', that is, a category for a different kind of units, called '***meta***-${}_c z$s', each of whose thus '***meta***-units', being made up out of a ***multiplicity*** of those units of which the category of the ${}_c z$s is made up.

To be ***specific*** with this rule, in our example-model, ${}_c\underline{q}_{zz}$ specifies a "**C**omplex" of two sub-categories.

Each of the units of the ***first*** sub-category, the sub-category of the '''***verse***''' operations, must be an '**Addition *OF* Additions**' that is, must be a '***meta-*Addition**', such that each '***meta-*Addition**' is made up out of a ***multiplicity*** of "mere" **Additions**.

Each of the units of the ***second*** sub-category, the sub-category of the '''***in**verse*''' operations, must be a '**Subtraction *OF* Subtractions**', that is, must be a '***meta-*Subtraction**', such that each such '***meta-*Subtraction**' is made up out of a ***multiplicity*** of "mere" **Subtractions**.

Our experiences in the domain of 'the basic operations of "**R**eal" arithmetic' suggest that such operations do "presently" exist in the domain of "**R**eal" arithmetic.

"Mu**L**tiplicatio**n**" is a basic arithmetical operation that is "made up out of multiple [repeated] additions", viz. --

$$4 \times 5 = 5 + 5 + 5 + 5 = 20 = 4 + 4 + 4 + 4 + 4 = 5 \times 4$$

-- a sum of four fives, or a sum of five fours: either order will do [a characteristic called "the commutativity of addition"]!

In a partial reverse likeness, "diVision" is a basic arithmetical operation that is "made up out of multiple [repeated] subtractions", viz. --

20 ÷ 5 = 4; 20 – 5 – 5 – 5 – 5 = 0 = 20 – 4 – 4 – 4 – 4 – 4; 20 ÷ 4 = 5

-- to see how many fours there are in twenty [¡***not*** the same as how many twenties there are in four!]; how many "times" four "goes ["evenly", i.e., with **0** "remainder"] in to" twenty, or to see how many fives there are in twenty, [[¡***not*** the same as how many twenties there are in five!]; how many "times" five "goes ["evenly"] in to" twenty: but, in this case, either order will ***not*** do!

A letter that the spelled names of these two kinds of operations have in common is "**n**".

Therefore, we shall name/symbolize our ***second*** category as $_c\mathbf{n}$, or as $_c\mathbf{q}_n$, denoting the "**C**omplex" combination of the "**muLtiplications**" sub-category of elementary **R**eal arithmetic basic operations, with the sub-category of "**diVisions**", and identifying that combination of ***specific*** sub-categories with the ***generic second*** category symbol of our ***generic*** category-arithmetic model, namely, with the ***generic*** category-symbol $_c\mathbf{q}_{[2 + 2i]}$.

We may "assert" our solution as follows:

$$_c\mathbf{q}_{tt} = {_c\mathbf{q}_n} \equiv {_c\mathbf{n}} \equiv {_c\mathbf{q}_{[L + Vi]}} = {_c\mathbf{q}_{[AA + SSi]}} \text{ [---) } {_c\mathbf{q}_{[2 + 2i]}}.$$

Again, what is ***dialectical*** about the relationship between $_c\mathbf{t}$ and $_c\mathbf{t}^2$, or $_c\mathbf{t} \times {_c\mathbf{t}}$, or $_c\mathbf{t}$ ***of*** $_c\mathbf{t}$, or $_c\mathbf{t}(_c\mathbf{t})$, the relationship of what we call '***meta-unit-ization***', or '***meta-«monad»-ization***', between $_c\mathbf{t}$ and its already presently existing, '**supplementary** other', $_c\mathbf{n}$, is that this relationship is a synchronic ***double-«aufheben»*** relationship.

That is, each single "unit" of the "**muLtiplications**" sub-category of category $_c\mathbf{n}$, i.e., each typical individual "multiplication" operation, is a ***negation***, and also a ***preservation***, by way of also being an ***elevation to** the* / ***forming*** *the* "higher" / more inclusive "**muLtiplications**" sub-category / level / scale, which is one of ***whole***, multiplicatively organized [*sub*-]***groups*** of repeating ***units*** from the "**Additions**" sub-category / level / scale of the $_c\mathbf{t}$ category.

Likewise, each single "unit" of the "**diVisions**" sub-category of category $_c\mathbf{n}$, i.e., each typical individual "division" operation, is a ***negation***, and also a ***preservation***, by way of also being an ***elevation to** the* / ***forming*** *the* "higher" / more inclusive "**diVisions**" sub-category / level / scale, of ***whole***, 'division-ally' organized [*sub*-]***groups*** of repeating ***units*** of the "**Subtractions**" sub-category / level / scale of the $_c\mathbf{t}$ category.

So, our full solution to the **s**tep **s = 1** equation of our model is --

$$_c\mathbf{)\text{-}|\text{-}(}_1 = {_c\mathbf{t}} + {_c\mathbf{n}} = {_c\mathbf{q}_{[A + Si]}} + {_c\mathbf{q}_{[L + Vi]}} =$$

Additions & Subtractions + **MuLtiplications & DiVisions**

$$\text{[---) } {_c\mathbf{q}_{[1 + 1i]}} + {_c\mathbf{q}_{[2 + 2i]}}.$$

If this model is working right, **Additions & Subtractions** is the ***simplest*** category of the domain of 'basic arithmetical operations', and **MuLtiplications & DiVisions** is the ***next more complex*** category of that domain.

Stage 2. ¿What additional 'category-*speci*fications' do we generate in our next step, **s = 2**, that need "solving-for"?

Let's find out:

$_c)\text{-}|\text{-}(_2 = {_c}t^{2^2} = {_c}t^4 = ({_c}t^2)^2 = ({_c}t + {_c}n)^2 = ({_c}t + {_c}n) \times ({_c}t + {_c}n) =$

${_c}t + {_c}n + {_c}q_{nt} + {_c}q_{nn}$.

This result arises by way of two key rules of dialectical categorial algebra, plus the ***general*** rule for multiplication when one category-symbol is multiplied by a different category-symbol [we used a ***special*** case of this ***general*** rule, for the case where the same category-symbol is multiplied by itself, in step **s = 1**, above] --

1. ***general*** case: ${_c}q_Y \times {_c}q_X = {_c}q_X + {_c}q_{YX} = {_c}X + {_c}q_{YX}$;

special case: ${_c}q_X \times {_c}q_X = {_c}q_X + {_c}q_{XX} = {_c}X + {_c}q_{XX}$.

2. ${_c}q_X + {_c}q_X = {_c}q_X$; the same category-symbol, added to itself, does not make "two" of that category-symbol; one "copy" of each category is sufficient; two or more copies of any category would be redundant for the purposes of this '***dialectical-categorial algebra***'.

3. There is no ${_c}q_W$ such that ${_c}q_X + {_c}q_Y = {_c}q_W$; *different* category-symbols, added together [as opposed to being '''multiplied'''], ***do not reduce*** to a single category-symbol, as in the proverbial case of '**apples + oranges**', or **a + o**.

Well, we already know how to "solve-for" ${_c}q_{nn}$.

It describes a category "containing" two sub-categories, the ***first*** sub-category being one of '**muLtiplications** ***OF*** **muLtiplications**', and the ***second*** sub-category being one of '**diVisions** ***OF*** **diVisions**'.

The ***first*** sub-category is one each of whose units / operations is a '**muLtiplication** ***OF*** **muLtiplications**', i.e., each of which is a '***meta***-**muLtiplication**', such that each such '***meta***-**muLtiplication**' operation is made up out of a multiplicity of **muLtiplication** operations.

Our experience of 'basic arithmetical operations' suggests that such arithmetical operations do indeed presently exist.

That sub-category-description describes the sub-category of '***multi***-**muLtiplication**' operations -- i.e., of "**exPonentiations**": "exPonentiation" is a basic arithmetical operation which is "made up out of multiple [repeated] **muLtiplication** operations, viz. --

$2^3 \equiv 2 \times\times 3 \equiv 2 \times 2 \times 2 = 8 \neq 9 = 3 \times 3 \equiv 3 \times\times 3 \equiv 3^2$.

I.e., "two cubed", or "two raised to the exponent three", is "made of" a product involving three twos, that yields eight, whereas "three squared", or "three raised to the exponent two", yields nine: in general, the order of "base" and "exponent" cannot be reversed without changing the result as well. Generally, each order will return a different result.

The ***second*** sub-category should be, per our standard method, one each of whose units / operations is a '**diVision** ***OF*** **diVisions**', i.e., each of which is a '***meta***-**diVision**', such that each such '***meta***-**diVision**' operation is made up out of a multiplicity of **diVision** operations. That is, the ***second*** sub-category should be one of '***multi***-**diVision**' operations, "made up out of multiple [repeated] **diVision** operations.

We interpret this to be the "inverse" operation of '***de***-exPonentiation', or of "**n**th **R**oot extraction".

The "log" operation, which returns exponents, not bases or roots, is also a candidate for this "inverse" operation, but is not as fully this inverse operation as is the **R**oot extraction operation.

Given our experience of 'basic arithmetical operations', this sub-category description may, at this point, give us pause.

Many of us may be unfamiliar with the algorithms by which the "**n**th" root(s) of a given number are "extracted". In what sense, if any, can an exponentiation be reversed, the "root" "extracted" from its "power", by repeated division?

Let us consider the method of extracting square roots that is perhaps the oldest such method still known. It is called "The Babylonian Method", and also "Heron's Method", because the storied Heron of Ancient Alexandria is the most ancient source known to have written an explicit account of this method. This method is, by the way, a ***speci***al case of the more ***gene***ral "Newton's Method", but predates the discovery of "Newton's Method" by many centuries. The method involves guessing a "***starting estimate***" for the square root sought, followed by repeated stages of, well, **diVision** -- division of the square by the current best estimate of its square root -- followed by, well, **diVision** again -- this time division of the sum of the previous consecutive pair of estimates by two, thus averaging them -- to obtain the next better estimate of the square root, all leading to an ever-improving estimate for the square root with each iteration of the '**double diVision**' just described, if the root is an "irrational" "**R**eal" number, "algebraic" or "transcendental", or is a never-repeating-decimal "Rational" "**R**eal" number, or to the "fixed point" or "equilibrium" of an ever-recurring answer if the root is a "**W**hole" "**R**eal" number, as in our example, below.

'Formulaically', the next better estimate of the square's square root, $\mathbf{x_{n+1}}$, is derived from the previous best estimate, $\mathbf{x_n}$, by **dividing** the square, **S**, by that previous best estimate, $\mathbf{x_n}$, summing $\mathbf{x_n}$ and $\mathbf{S \div x_n}$, then **dividing** that sum by **2**:

$$x_{n+1} \;=\; (x_n + S \div x_n) \div 2.$$

Let us apply this method to "extracting" the "square root" from the "square", **9**, with "***starting estimate***" of $\mathbf{x_1 = 2}$ --

n	Current Best Estimate ($\mathbf{x_n}$)	**DiVide** Square by That Estimate ($\mathbf{S \div x_n}$)	**DiVide** Their Sum by **2** for new best est.
1	**2**	$\mathbf{9 \div 2 = 4.5}$	$\mathbf{(2+4.5) \div 2 = 3.25}$
2	**3.25**	$\mathbf{9 \div 3.25 \approx 2.769}$	$\mathbf{(3.25+2.769) \div 2 \approx 3.01}$
3	**3.01**	$\mathbf{9 \div 3.01 \approx 2.99}$	$\mathbf{(3.01+2.99) \div 2 \approx 3.000}$
4	**3.000**	$\mathbf{9 \div 3.000 = 3.000}$	$\mathbf{(3.000+3.000) \div 2 = 3.000}$

After **n** = **3**, for any **n** > **3** that you try, with the rounding as shown above, the method produces a "fixed point" /-"equilibrium" at $\mathbf{x_{3+\ldots} = 3}$, which ***is*** the positive square root of **9**.

Thus we see in what sense, in this method at least, square root extraction is made up out of repeated **diVision**s.

We may thus now "assert" our solution as follows --

$$_{c}\mathbf{q}_{nn} \;=\; _{c}\mathbf{q}_{e} \;\equiv\; _{c}\mathbf{e} \;\equiv\; _{c}\mathbf{q}_{[P + Ri]} \;=\; _{c}\mathbf{q}_{[LL + vvi]} \;\; [\text{---}) \;\; _{c}\mathbf{q}_{[4 + 4i]}.$$

Our **s**tep **s = 2** equation-model, as we have solved it so far, thus now looks like this --

$$_{c}\mathbf{)\text{-}|\text{-}(}_{2} \;=\; _{c}\mathbf{t}^{2^{2}} \;=\; _{c}\mathbf{t}^{4} \;=\; _{c}\mathbf{t} + _{c}\mathbf{n} + _{c}\mathbf{q}_{nt} + _{c}\mathbf{e}$$

$$[\text{---}) \;\; _{c}\mathbf{q}_{[1 + 1i]} + _{c}\mathbf{q}_{[2 + 2i]} + _{c}\mathbf{q}_{[3 + 3i]} + _{c}\mathbf{q}_{[4 + 4i]}$$

-- since we have not yet determined which actual category of the 'basic arithmetic operations' domain is described by the algorithmically-generated symbol $_{c}\mathbf{q}_{nt}$ -- if any, i.e., if $_{c}\mathbf{q}_{nt}$ is not an "empty category", "inoperative" for this domain.

When, as a component of $({}_{c}\mathbf{t} + {}_{c}\mathbf{n}) \times ({}_{c}\mathbf{t} + {}_{c}\mathbf{n})$, the "higher-complexity" / 'more-inclusive' category, ${}_{c}\mathbf{n}$, operates upon / "multiplies" the "lower-complexity" / 'less-inclusive' category, ${}_{c}\mathbf{t}$ --

$$ {}_{c}\mathbf{n} \times {}_{c}\mathbf{t} = {}_{c}\mathbf{t} + {}_{c}\mathbf{q}_{\mathbf{nt}} = {}_{c}\mathbf{q}_{[\mathbf{A} + \mathbf{Si}]} + {}_{c}\mathbf{q}_{[\mathbf{LA} + \mathbf{VSi}]} $$

-- ***generically*** speaking, the categorial relationship to be called to the user's attention by this operation, in this 'categorial algebra', is, again, a synchronic «***aufheben***» relationship, this time, that between ${}_{c}\mathbf{t}$ and ${}_{c}\mathbf{q}_{\mathbf{nt}}$.

It calls the user to search that user's knowledge and memory of the domain in question -- in this ***specific*** case, the domain of 'basic arithmetical operations' -- for a category which represents an "uplift" of category ${}_{c}\mathbf{t}$ entities to the level of the entities native to category ${}_{c}\mathbf{n}$, thereby "***canceling***" the ${}_{c}\mathbf{t}$ -type entities concerned, at their own native level, but, by the same token, "***preserving***" those category ${}_{c}\mathbf{t}$ entities at the ${}_{c}\mathbf{n}$ level, ***combining*** ${}_{c}\mathbf{n}$ and ${}_{c}\mathbf{t}$ **q**ualities, in the relationship of "***elevation***" of those category ${}_{c}\mathbf{t}$ entities up to within the level typical of category ${}_{c}\mathbf{n}$ entities. Thus, the additional category thereby presented, ${}_{c}\mathbf{q}_{\mathbf{nt}}$, signifies a category whose units are the ***operational interactions*** of the ${}_{c}\mathbf{t}$ operations with the ${}_{c}\mathbf{n}$ operations, as codified in the axioms, and/or theorems, and/or corollaries, and/or lemmas, and/or '''rules''' of the "**R**eal Numbers" system of arithmetic.

The ***first*** sub-category of category ${}_{c}\mathbf{q}_{\mathbf{nt}} = {}_{c}\mathbf{q}_{[\mathbf{LA} + \mathbf{VSi}]}$ answers to a sub-category description for the way in which, or for the '''rules''' by which, the operation of **muLtiplication** "***subsumes***" that of **Addition**, denoted herein by '**L | A**'.

To our lights, this sub-category-description connotes the "distributive law" of "**R**eal" arithmetic, an axiom of that system of arithmetic, which codifies the interaction of the addition operation with the multiplication operation -- the rule that the multiplication operation "distributes over" the addition operation. This "law" involves two components, often called "left distributivity" and "right distributivity", respectively:

- ["left distributivity"]: For all elements **a**, **b**, and **c** of **R**, $\mathbf{c \times (a + b) = (c \times a) + (c \times b)}$.
- ["right distributivity"]: For all elements **a**, **b**, and **c** of **R**, $\mathbf{(a + b) \times c = (a \times c) + (b \times c)}$.

The ***second*** sub-category of the category ${}_{c}\mathbf{q}_{\mathbf{nt}} = {}_{c}\mathbf{q}_{[\mathbf{LA} + \mathbf{VSi}]}$ answers to a sub-category description which connotes the '''rules''' by which the operation of **diVision** "subsumes" the operation of **Subtraction**, denoted '**V | S**'.

To our lights, this sub-category-description connotes a "***non***-distributive rule" of "**R**eal" arithmetic for '**diVision** / **Subtraction**', although this rule is, typically, not an explicit one in presentations and in axiomatizations of "**R**eal" arithmetic. It is learned informally, as a joint consequence of other rules, i.e., as [partly] already subsumed under, or included in, the "distributive law", or is encountered as a theorem, corollary, or lemma. First of all, note that **diVision** does ***not*** fully "distribute" over ['**|**'] **Addition**:

- ['''left ***non***-distributivity''', '**V | A**']: For some **a**, **b**, **c** of **R**, $\mathbf{(a + b) \neq 0}$, $\mathbf{c \div (a + b) \neq (c \div a) + (c \div b)}$.
- ['''right distributivity''', '**V | A**']: For all **a**, **b**, **c** of **R**, $\mathbf{c \neq 0}$, $\mathbf{(a + b) \div c = (a \div c) + (b \div c)}$.

The $\mathbf{(a + b) \neq 0}$ and $\mathbf{c \neq 0}$ proviso's are necessary, in these assertions about '**V | A**', because **diVision**s by zero invoke a value that resides beyond the '''number-space''' of the set **R**.

But the ***second*** sub-category of $_{c}\mathbf{q}_{nt} = {}_{c}\mathbf{q}_{[LA + VSi]}$ pertains directly to the interaction of the **diVision** operation with the **Subtraction** operation, not with the **Addition** operation [although, given that the set **R** includes "signed numbers" with "negative" signs, i.e., "additive inverses", as well as the subtraction operation-sign, additions can also express subtractions, i.e., if **b = –d**, then **a + b = a – d**], e.g.:

- ['''left ***non***-distributivity''', '**V | S**']: For some **a**, **b**, **c** of **R**, **(a – b) ≠ 0**, **c ÷ (a – b) ≠ (c ÷ a) – (c ÷ b)**, e.g., **3 ÷ (1 – 2) = –3 ≠ 1.5 = (3÷1) – (3÷2)**.

- ['''right distributivity''', '**V | S**']: For all **a**, **b**, **c** of **R**, **c ≠ 0**, **(a – b) ÷ c = (a ÷ c) – (b ÷ c)**.

If we re-express subtractions as additions, and divisions as multiplications -- which the "**R**eal" number arithmetic enables us to do, since it includes ratios and "multiplicative inverses", as well as "additive inverses", we see that the "**R**eal" arithmetic's rules for '**V | S**' are ***partly*** implicit in the '**L | A**' rules, e.g., if we set **b = –d** and **c = 1÷e ≡ 1/e**, **e ≠ 0**:

- [an aspect of '**L | A**']: For **a**, **-d**, **1/e** of **R**, **(a + d) ≠ 0**, **1/e x 1/(a + d) = 1/((e x a) + (e x b))**, e.g., **1/(1/3) x 1/(1 + –2) = –3 = 1/(1/3 x 1) + (1/3 x –2))**.

- ['''right distributivity''', '**L | A**']: For all **a**, **-d**, **1/e** of **R**, **e ≠ 0**, **(a + d) x 1/e = (a x 1/e) + (d x 1/e)**, e.g., **(1 + –2) x (1/(1/3)) = –3 = (1 x (1/(1/3))) + (–2 x (1/(1/3)))**.

The subscript '$_{VSi}$' component of the subscript '$_{[LA + VSi]}$' of category-symbol $_{c}\mathbf{q}_{[LA + VSi]} = {}_{c}\mathbf{q}_{nt}$ can thus be interpreted as calling attention ***systematically*** and explicitly, if *somewhat* redundantly, to the ***speci***fic '**V | S**' rules, which differ from the ***gene***ric '**L | A**' rules, in that the '**V | S**' rules require the making explicit of ***speci***al restrictions [e.g., **0** denominators not allowed], etc., as we have seen above.

We may therefore write our *full* solution for step **s = 2** as --

$$_{c})\text{-|-}(_{2} \;=\; {}_{c}\mathbf{t}^{2^{2}} \;=\; {}_{c}\mathbf{t}^{4} \;=\; {}_{c}\mathbf{t} + {}_{c}\mathbf{n} + {}_{c}\mathbf{q}_{nt} + {}_{c}\mathbf{e}$$

Additions & Subtractions + **muLtiplications & diVisions** + **n & t interactions** +

exPonentiations & *de*-exponentiations.

The 'qualo-fractal' content-structure of this psychohistorical dialectic to step **2** can be summarized as follows --

exPonentiations & *de*-exponentiations "contain" **muLtiplications & divisions**,

which, in turn, "contain" **Additions & Subtractions**.

The "five symbolic-elements expression" for this model is thus $_{c}\mathbf{t}^{2^{2}}$ [five if we count the underscore under the **t** as a separate "symbolic-element"].

The systematic dialectic of the basic operations of arithmetic '''presently''' and "standardly" ends here, at step **s = 2**, with the category named **exponentiations & *de*-exponentiations** as 'meta-meristemal' or '''vanguard''' category.

We like the compactness of the $_{c}\mathbf{Q}$ model of this ***systematic dialectic***, whose "final step" requires only a single equation:

$$_{c})\text{-|-}(_{2} \;=\; {}_{c}\mathbf{t} + {}_{c}\mathbf{n} + {}_{c}\mathbf{q}_{nt} + {}_{c}\mathbf{e}.$$

Alternatively, we can express each **s**tep of this ***dialectic*** by ***two separate equations***, using the $_{N}$**Q** ***dialectical algebra***:

$$_{verse})\text{-|-}(_2 \;=\; {_c}\mathbf{A} + {_c}\mathbf{L} + {_c}\mathbf{q}_{LA} + {_c}\mathbf{P}$$, for the ‘‘‘*verse*’’’ operations.

$$_{inverse})\text{-|-}(_2 \;=\; {_c}\mathbf{S} + {_c}\mathbf{V} + {_c}\mathbf{q}_{VS} + {_c}\mathbf{R}$$, for the ‘‘‘***in**verse*’’’, or ‘‘‘***re**verse*’’’, operations.

Stage 3. To iterate of our $_c$**Q** ‘meta-equation’, $_c)\text{-|-}(_s \;=\; {_c}\mathbf{t}^{2^s} \;\equiv\; ({_c}\mathbf{q}_{[A + Si]})^{2^s}$, for **s**tep $s = 3$, is to iterate the systematic presentation of the domain of basic arithmetical operations beyond the “basic”, beyond the “present”, beyond the conventional conclusion of that presentation, and beyond the “systematic reconstruction” of this domain at present, to a somewhat “preconstructive” -- somewhat “predictive” -- extrapolation of its possible future. However, as we shall see, we have already encountered units of the “vanguard” term of **s**tep $s = 3$, ***in this very text***.

Let’s see what are the additional category-descriptions that this **s**tep $s = 3$ ‘self-iteration’ ***gene***rates:

$$_c)\text{-|-}(_3 \;=\; {_c}\mathbf{t}^{2^3} \;=\; {_c}\mathbf{t}^{8} \;=\; ({_c}\mathbf{t}^{4})^2 \;=\; ({_c}\mathbf{t} + {_c}\mathbf{n} + {_c}\mathbf{q}_{nt} + {_c}\mathbf{e})^2 \;=$$

$$({_c}\mathbf{t} + {_c}\mathbf{n} + {_c}\mathbf{q}_{nt} + {_c}\mathbf{e}) \times ({_c}\mathbf{t} + {_c}\mathbf{n} + {_c}\mathbf{q}_{nt} + {_c}\mathbf{e}) \;=$$

$$ {_c}\mathbf{t} + {_c}\mathbf{n} + {_c}\mathbf{q}_{nt} + {_c}\mathbf{e} + {_c}\mathbf{q}_{et} + {_c}\mathbf{q}_{en} + {_c}\mathbf{q}_{ent} + {_c}\mathbf{q}_{ee}$$

[---)

$$ {_c}\mathbf{q}_{[1 + 1i]} + {_c}\mathbf{q}_{[2 + 2i]} + {_c}\mathbf{q}_{[3 + 3i]} + {_c}\mathbf{q}_{[4 + 4i]} +$$

$$ {_c}\mathbf{q}_{[5 + 5i]} + {_c}\mathbf{q}_{[6 + 6i]} + {_c}\mathbf{q}_{[7 + 7i]} + {_c}\mathbf{q}_{[8 + 8i]}.$$

We know -- from past experience, narrated above -- how to “solve-for” category $_c\mathbf{q}_{ee} \;=\; {_c}\mathbf{q}_{[PP + RRi]}$.

It describes a category “containing” two sub-categories, the ***first*** sub-category being one of ‘**Powers *OF* Powers**’, and the ***second*** sub-category being one of ‘**Root-extractions *OF* Root-extractions**’.

The ***first*** sub-category is one each of whose units / operations is an ‘**exPonentiation *OF*** *an* **exPonentiation**’, i.e., each of which is a ‘***meta*-exponentiation**’, such that each such ‘***meta*-exPonentiation**’ operation is made up out of a multiplicity of **exPonentiation** operations. ***But that is precisely the new operation that we have encountered in this text***, at the heart of the Seldon Functions in ***gene*ral**, and at the heart of our $_c$**Q** ‘meta-equation’, $_c)\text{-|-}(_s \;=\; {_c}\mathbf{t}^{2^s}$, ***speci*fically**.

A unit increment in the ‘meta-exponent’ of the ‘***starting-category***’ symbol of that ‘meta-equation’, corresponding to a unit increment in its **s**tep-value, **s**, is equivalent to a two-fold exponentiation of that ‘***starting-category***’ symbol, e.g. --

$$ {_c}\mathbf{t}^{2} = {_c}\mathbf{t}^{2^1};$$

$$({_c}\mathbf{t}^{2})^{2} \;=\; ({_c}\mathbf{t}^{2^1})^{2} \;=\; ({_c}\mathbf{t}^{2^1})^{2^1} \;=\; {_c}\mathbf{t}^{2^{1+1}} \;\equiv\; {_c}\mathbf{t}^{2^2}$$

-- because repeated exponents mutually ***multiply***, and because ‘*meta-exponents*’ *of* exponents ***add*** together when those ‘meta-exponents’ have, as their bases, the same exponents, and when those ‘exponented exponents’ multiply together.

Let's call this sub-category **Hyper-exponentiation**, or **H** for short. The ***second*** sub-category should be for operations which are '**de-exponentiation**s ***OF*** **de-exponentiation**s', i.e., which are '***meta***-**de-exponentiation**s', such that each '***meta***-**de-exponentiation**' operation is made up out of a multiplicity of **de-exponentiation** operations, denoted by '√'.

That is, the ***second*** sub-category should be one of '***multi***-**de-exponentiation**' operations, "made up out of multiple [repeated] ***de*-exponentiation** operations. The 'self-example' of this very text illustrates this process:

$$\sqrt{\sqrt{\sqrt{{}_c\mathbf{t} + {}_c\mathbf{n} + {}_c\mathbf{q}_{nt} + {}_c\mathbf{e} + {}_c\mathbf{q}_{et} + {}_c\mathbf{q}_{en} + {}_c\mathbf{q}_{ent} + {}_c\mathbf{q}_{ee}}}} = \sqrt{\sqrt{\sqrt{{}_c\mathbf{t}^{2^3}}}} =$$

$$\sqrt{\sqrt{{}_c\mathbf{t} + {}_c\mathbf{n} + {}_c\mathbf{q}_{nt} + {}_c\mathbf{e}}} = \sqrt{\sqrt{{}_c\mathbf{t}^{2^2}}} =$$

$$\sqrt{{}_c\mathbf{t} + {}_c\mathbf{n}} = \sqrt{{}_c\mathbf{t}^{2^1}} =$$

${}_c\mathbf{t}^{2^{3-3}} = {}_c\mathbf{t}^{2^0} = {}_c\mathbf{t}$. Let's call this sub-category ***De*-Hyper-exponentiation**, or **D** for short.

We may "assert" our solution as follows: ${}_c\mathbf{q}_{ee} = {}_c\mathbf{q}_{m} \equiv {}_c\mathbf{m} \equiv {}_c\mathbf{q}_{[H + Di]}$ **[---)** ${}_c\mathbf{q}_{[8 + 8i]}$.

Our **s**tep **s = 3** equation-model, as we have solved it so far, thus now looks like this --

$${}_c\mathbf{)\text{-}|\text{-}(}_3 = {}_c\mathbf{t}^{2^3} = {}_c\mathbf{t}^{8} = {}_c\mathbf{t} + {}_c\mathbf{n} + {}_c\mathbf{q}_{nt} + {}_c\mathbf{e} + {}_c\mathbf{q}_{et} + {}_c\mathbf{q}_{en} + {}_c\mathbf{q}_{ent} + {}_c\mathbf{m}$$

-- since we have not yet determined which actual categories of the 'basic arithmetical operations' domain are described by the algorithmically-generated 'category-description' symbols ${}_c\mathbf{q}_{et}$, ${}_c\mathbf{q}_{en}$, and ${}_c\mathbf{q}_{ent}$, ***if any***. But we already know how to characterize the ***possible*** categories that these three category-symbols "call for", viz.:

- ${}_c\mathbf{q}_{et}$ **[---)** ${}_c\mathbf{q}_{[5 + 5i]}$ "calls for" a category for a kind of '''hybrid''' 'meta-operation', or 'operation of operations', that combines the **e** & **t** operations.
- ${}_c\mathbf{q}_{en}$ **[---)** ${}_c\mathbf{q}_{[6 + 6i]}$ "calls for" a category for a kind of '''hybrid''' 'meta-operation', or 'operation of operations', that combines the **e** & **n** operations.
- ${}_c\mathbf{q}_{ent}$ **[---)** ${}_c\mathbf{q}_{[7 + 7i]}$ "calls for" a category for a kind of '''hybrid''' 'meta-operation', or 'operation of operations', that combines the **e** & the ${}_c\mathbf{q}_{nt}$.

We may thus write our *full* solution for step **s = 3** as --

$${}_c\mathbf{)\text{-}|\text{-}(}_3 = {}_c\mathbf{t}^{2^3} = {}_c\mathbf{t}^{8} = {}_c\mathbf{t} + {}_c\mathbf{n} + {}_c\mathbf{q}_{nt} + {}_c\mathbf{e} + {}_c\mathbf{q}_{et} + {}_c\mathbf{q}_{en} + {}_c\mathbf{q}_{ent} + {}_c\mathbf{m} =$$

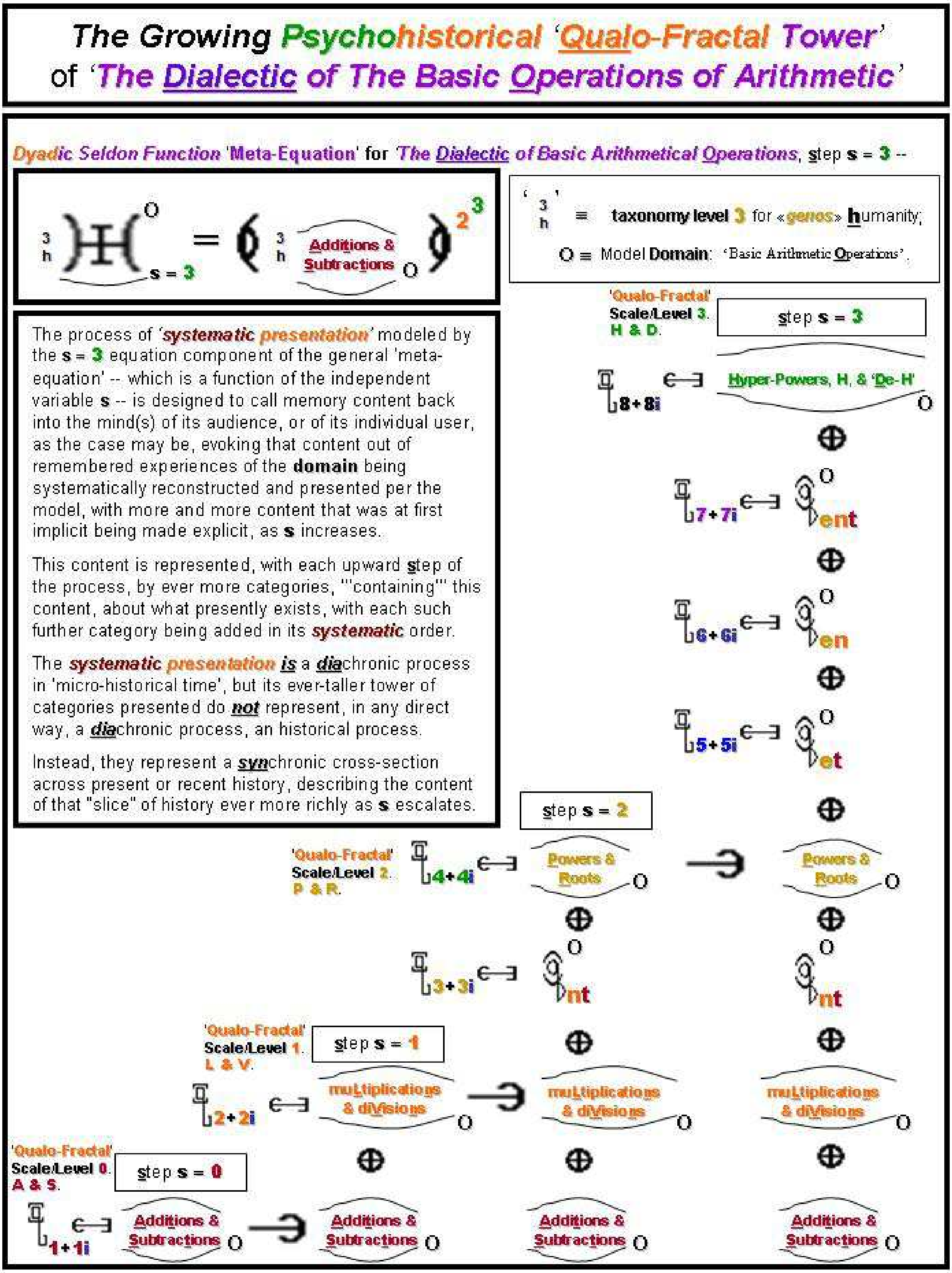
Additions & Subtractions + **muLtiplications & diVisions** + **n with t interactions** +
exPonentiations & *de*-exponentiations +
e with t interactions + **e with n interactions** + **e with n & t interactions** +
meta-exponentiations & D*e*-meta-exponentiations.

The 'qualo-fractal' content-structure of this psychohistorical dialectic through **s**tep **3** can be summarized as follows --

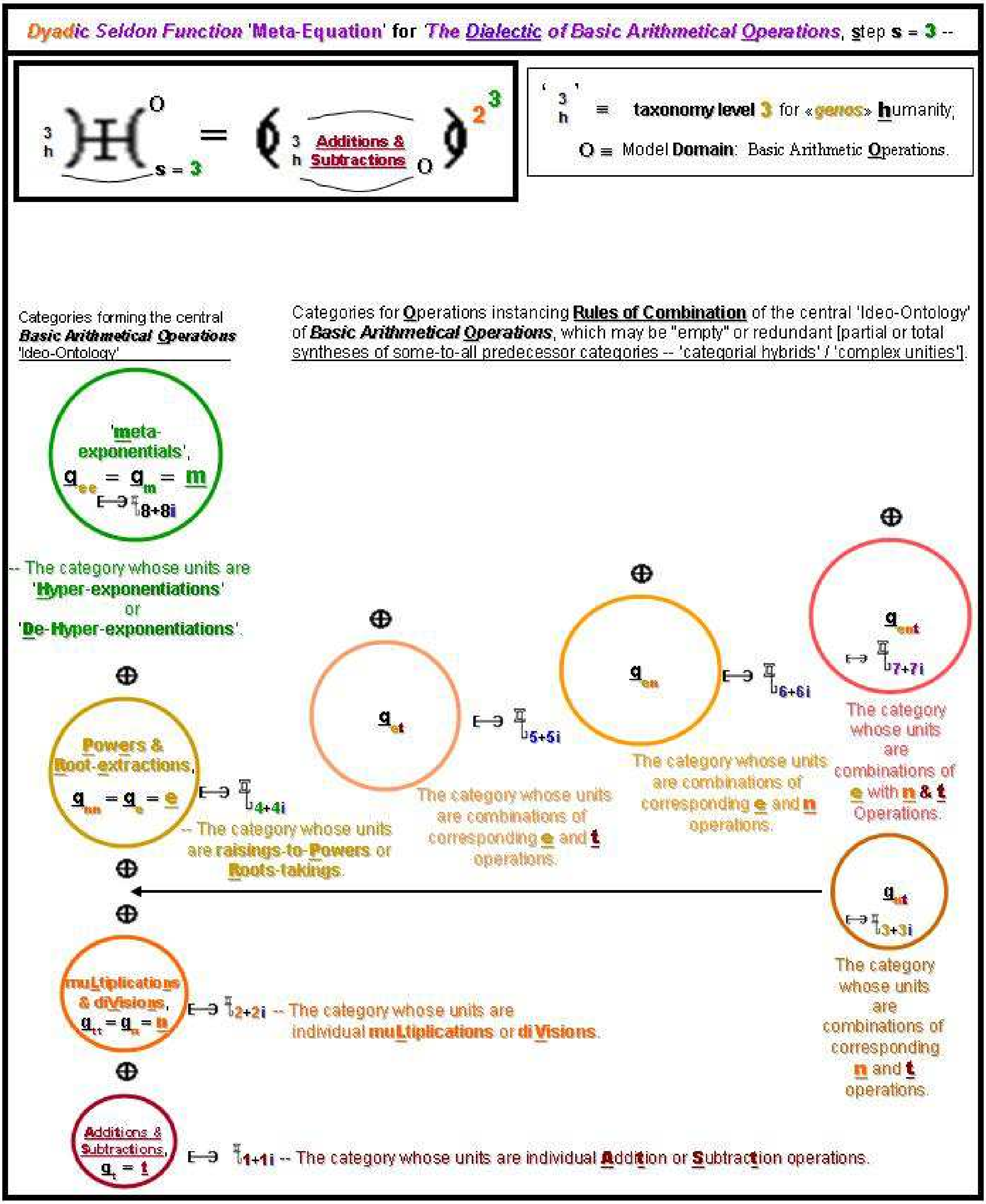
meta-exponentiations & De-meta-exponentiations "contain" **exPonentiations & *de*-exponentiations**, which, in turn, "contain" **muLtiplications & diVisions**, which, in turn, "contain" **Additions & Subtractions**.

The "five symbolic-elements expression" for this model, for this **s**tep, is thus ${}_c\mathbf{t}^{2^3}$.

The meaning mnemonically compressed into the five symbolic-element expression $_{c}\underline{t}^{2^{3}}$ can be depicted as follows --



Psychohistorical Ontological Categories Diagram for '*The Dialectic of The Basic Operations of Arithmetic*'



Links to definitions of additional ***Encyclopedia Dialectica*** special terms also involved in the discourse above --

«***arithmos aisthetos***»
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/ArithmosAisthetos/ArithmosAisthetos.htm

«***arithmos eidetikos***»
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/ArithmosEidetikos/ArithmosEidetikos.htm

categorial
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Categorial/Categorial.htm

category
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Category/Category.htm

'***consecuum***'
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Consecuum/Consecuum.htm

'***cumulum***'
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Cumulum/Cumulum.htm

dialectical categorial progressions
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/CategorialProgression/CategorialProgression.htm

homeomorphic defects of models
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/HomeomorphicDefect/HomeomorphicDefect.htm

[***The***] ***Human Phenome***
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/PsychoHistory/PsychoHistory.htm

immanent
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Immanent/Immanent.htm

immanent critique
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/ImmanentCritique/ImmanentCritique.htm

«***monad***»
https://www.point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Monad/Monad.htm

ontological category
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/CategoryOntological/CategoryOntological.htm

ontology
https://www.point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Ontology/Ontology.htm

psychohistory
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/PsychoHistory/PsychoHistory.htm

qualo-fractal
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/QualoFractal/QualoFractal.htm

qualo-Peanic
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/QualoPeanicity/QualoPeanicity.htm

'***self-meta-monad-ization***' or '***self-meta-individual-ization***' or '***self-meta-holon-ization***'
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/Meta/Meta.htm
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/MetaMonadization/MetaMonadization.htm

"solution procedure": ***the organonic algebraic method for solving dialectical*** [***meta-***]***equations***
http://point-of-departure.org/Point-Of-Departure/ClarificationsArchive/OrganonicAlgebraicMethod/OrganonicAlgebraicMethod.htm