Discovering Integer-Qualifier Space ($_{Z}\underline{Q}$) via Z-Cum Space ($\underline{C}_Z$)

(plus an alternative $_{Z^*}\underline{Q}$ and $\underline{C}_{Z^*}$ based on Z^* := "2-D Whole Numbers")

by

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*About **E.D. Brief #7***: This article is meant to extend the **W-Cum** ($\underline{C}_W$) and $_W\underline{Q}$ co-discoveries made in Briefs **#5** and **#6** to co-discoveries of **Z-Cum** ($\underline{C}_Z$) and $_Z\underline{Q}$. By answering, "Is $\underline{C}_1 + (\underline{C}_1)^{-1} = C_0$?", it is possible to construct two versions of **Z-Cum** space, one for which $\underline{q}_n + \underline{q}_{(-n)} = q_0$ (the F.E.D.$^{++}$ formulation) based on the Integers (**Z**), and a version which assumes that $\underline{q}_z + \underline{q}_{(-z)}$ is not q_0, but rather is a "non-amalgamative sum" not equal to q_0 or to any $\underline{q}_{Z^*}$. This later **Z*-Cum** space is based on perhaps a new quantitative system called the "**2-D W***hole Numbers*" ($W^{2\text{-}D}$), developed in an appendix. Both formulations share the common **Cum-×**: $\underline{C}_k \times \underline{C}_n = \underline{C}_{k+n}$ (for any **W**hole **k**, **n**), which is the basis for expanding the **W-Cum** space to the **Z-Cum** space. *The meta-genealogical product rule* is employed in both versions. In an appendix, proof is offered regarding possible meanings for $\underline{q}_k$^$\underline{q}_n$ for **k**, **n** in **{–1, ±0, +1}**.

Note to Reader on Prerequisite Briefs: Our co-discovery herein crucially depends upon the reader being familiar with ***E.D. Brief #5*** & ***E.D. Brief #6***. We urge readers to read those briefs before reading this one.

1. Overview and Question: "Is a 'nullifying cumulation' possible?"

In ***E.D. Brief #5***, "Discovering **N**atural-**Q**ualifiers Space ($_N\underline{Q}$) via **N-Cum** ($\underline{C}_N$) Space", an isomorphic map, $\text{exQ}(n) := (\underline{C}_1)^n = \underline{C}_n$, is used to map the **N**aturals onto **N-Cum** ($\underline{C}_N$), a "**C**umulation space" of idea set-numbers. In ***E.D. Brief #6***, this map was extended, via **0**, to the *origin cumulation,* $C_0 := \text{exQ}(0) := (\underline{C}_1)^0$, which is the same as the *origin* **q***ualifier*: $q_0 := C_0$ -- identically the ***additive identity element***, and the ***multiplicative identity element***, in both the $_W\underline{Q}$ space and the **W-Cum** ($\underline{C}_W$) space.

In this brief, in our process of co-discovering versions of "Integer **C**umulation" space with the reader, we first ask these new "early questions": *"Is it possible to nullify or reverse an idea?* I.e., "Is it possible to have an idea set, **X**, which can bring an existing idea-cumulation, $\underline{C}_n$, back to a previous one, say $\underline{C}_{n-1}$?" Mathematically, we are asking if there can exist an **X**, such that $\underline{C}_n \times X = \underline{C}_{n-1}$? This leads to the questions: *"Can we have an inverse cumulation,* $(\underline{C}_n)^{-1}$, of $\underline{C}_n$, so that $(\underline{C}_n)^{-1} \times \underline{C}_n := C_0$*?"* and if so, *"What is the nature of the possible resulting space?"*

Before getting into the possible mathematics, let's discuss this very notion of a 'nullifying' cumulation. In our everyday parlance, we might hear such utterances as: *"He wants to take us back to a time when…"* or "*Her ideas about…are reactionary!*". Indeed, there seems to be the possibility that certain ideas could "reverse progress", so to speak. But this does not mean we can actually go back in time, as is pointed out in **F.E.D. Vignette #2**: ***Time Actual***. We merely proceed in epochal time τ (not necessarily *uniform calendar/clock time*) with the possibility that in epoch $\tau+1$, we could, nonetheless, have

$$(\underline{C}_{\tau+1})' = \underline{C}_\tau \times X = \underline{C}_{\tau-1}?$$

So, it is reasonable to pursue such a "possibility extension" of our cumulation space, the **W-Cum** space, and of its **q**ualifiers.

2. Co-Discovering a "**Z**-$\underline{\text{C}}$**um** Space"

We begin where we left off in ***Brief #6***, with the **W**hole $\underline{\text{C}}$**um**s, $\underline{C}_{\mathbf{W}}$. First, we consider solving $\underline{C}_{n-1} = \underline{C}_n \times \underline{X}$ for some $\underline{X} = \underline{C}_x$. Assuming that our extended $\times$ behaves as in $\underline{C}_{\mathbf{W}}$, we have $\underline{C}_{n-1} = \underline{C}_n \times \underline{C}_x = \underline{C}_{n+x}$, which implies that $n-1 = n + x$, or $x = -1$. Thus, $\underline{X} = \underline{C}_{-1}$ and $\underline{C}_{-1} \times \underline{C}_1 = C_{0+0} = C_0$, or $\underline{C}_{-1}$ is the $\underline{\text{C}}$**um** $\times$ inverse of $\underline{C}_1$! And we suspect that $(\underline{C}_{-1})^{w}$ might give birth to an alternative **W**-$\underline{\text{C}}$**um** ($\underline{C}_{\mathbf{W}}$), just as $(\underline{C}_1)^{w}$ generated $\underline{C}_{\mathbf{W}}$. With our "preliminary theorizing" done, we now postulate that such an $\underline{X} = \underline{C}_x$ exists:

Initial Nullifier Existence Postulate: There exists a *nullifier cumulation*, $(\underline{C}_{+1})^{-1} := \underline{C}_{1^*} := \underline{C}_{-1}$, with $\underline{C}_{-1} \times \underline{C}_{+1} = C_{\pm 0}$. [implies **Z**-$\underline{\text{C}}$**um** exists as: $\{(\underline{C}_{-1})^{z}\!: \; z \text{ in } \mathbf{Z}\} := \underline{C}_{\mathbf{Z}} := \{(\underline{C}_{+1})^{z}\!: \; z \text{ in } \mathbf{Z}\}$.]

Next, we make an isomorphic copy of $\underline{C}_{\mathbf{W}}$ to create a "complementary $\underline{\text{C}}$**um** space," say **W***-$\underline{\text{C}}$**um** or $\underline{C}_{\mathbf{W}^*} := \{\underline{C}_{w^*}\!: w^* \text{ in some } \mathbf{W}^* \text{ set complementary to } \mathbf{W}\}$. If $\underline{C}_{w^*}$ in $\underline{C}_{\mathbf{W}^*}$ were to be a "**Cum** $\times$ inverse" of $\underline{C}_w$ in $\underline{C}_{\mathbf{W}}$ under the same type of $\underline{\text{C}}$**um** $\times$ extended to $\underline{C}_{\mathbf{W}} \cup \underline{C}_{\mathbf{W}^*}$, then $\underline{C}_w \times \underline{C}_{w^*} := \underline{C}_{w+w^*} := C_0$, so we might claim that $w + w^* = 0$, or that $w^* = -w$. This would say that our complement of $\mathbf{W}$, $\mathbf{W}^* := \{-w\!: \; w \text{ in } \mathbf{W}\}$. Thus, our expanded/extended $\underline{\text{C}}$**um** space *appears* to be:

$$\underline{C}_{\mathbf{Z}} := \underline{C}_{\mathbf{W}} \cup \underline{C}_{\mathbf{W}^*} = \underline{C}_{\mathbf{Z}} := \underline{C}_{\mathbf{W}} \cup \underline{C}_{-\mathbf{W}} := \underline{C}_{\mathbf{W} \cup (-\mathbf{W})} := \underline{C}_{\mathbf{Z}}.$$

In other words, the $\underline{\text{C}}$**um** space that would contain the $\underline{\text{C}}$**um** $\times$ inverses of $\underline{C}_{\mathbf{W}}$ is:

$$\underline{C}_{\mathbf{Z}} := \text{Integer } \underline{\text{C}}\textbf{um} \text{ space} = \mathbf{Z}\text{-}\underline{\text{C}}\textbf{um} \text{ space!}$$

Because $\underline{C}_{-1}$ is the isomorphic image of $\underline{C}_{+1}$, $\underline{C}_{-1}$ generates $\underline{C}_{\mathbf{Z}} := (\underline{C}_{-1})^{z}$ in a similar way to the way $\underline{C}_{+1}$ generates $\underline{C}_{\mathbf{Z}} := (\underline{C}_{+1})^{z}$. And, as $\underline{C}_{-1}$ is defined to be the multiplicative inverse of $\underline{C}_{+1}$, we have --

$$\underline{C}_z := [\underline{C}_{-1}]^{z} = [(\underline{C}_{+1})^{-1}]^{z} := (\underline{C}_{+1})^{(-z)}.$$

We now see that for integer exponents (i.e., any $+z$ or $-z$ in $\mathbf{Z}$), either $\underline{C}_{+1}$ or $\underline{C}_{-1}$ alone, under $\underline{\text{C}}$**um** $\times$, can generate all of **Z**-$\underline{\text{C}}$**um** space! Expressed in **F**.***E***.***D***. terms: "$\underline{C}_{+1}$" is the «***arché***»/"base" that generates **Z**-$\underline{\text{C}}$**um**, and so is "$\underline{C}_{-1}$" -- an "alternate" «***arché***»/"base", that also generates **Z**-$\underline{\text{C}}$**um**.

3. Deriving Integer-Qualifier Space, $_{\mathbf{Z}}\underline{Q}$

The corresponding Integer-$\underline{\text{Q}}$ualifier space, $_{\mathbf{Z}}\underline{Q}$, can be defined as the set of differentials of all **Z**-$\underline{\text{C}}$**um**s, or as the union of the $\underline{\text{q}}$ualifier spaces that correspond to the **W**-$\underline{\text{C}}$**um** and **W***-$\underline{\text{C}}$**um** spaces:

$$_{\mathbf{Z}}\underline{Q} := \{\underline{q}_z := \partial\underline{C}_z\!: \text{ for all integers } z \; (z \text{ in } \mathbf{Z})\}, \text{ or}$$

$$_{\mathbf{Z}}\underline{Q} := {_{\mathbf{W}}\underline{Q}} \cup {_{\mathbf{W}^*}\underline{Q}} := \{\underline{q}_w := \partial\underline{C}_w\!: \text{ for all } w \text{ in } \mathbf{W}\} \cup \{\underline{q}_{-w} := \partial\underline{C}_{-w}\!: \text{ for all } w \text{ in } \mathbf{W}\}.$$

But now we must inquire: "Is each succeeding $\underline{q}_{z+1}$ (or preceding $\underline{q}_{-z-1}$) qualitatively more (or less) definite than the previous $\underline{q}_k$?" To answer this, we have two separate $_{\mathbf{W}}\underline{Q}$ and $_{\mathbf{W}^*}\underline{Q}$ element orderings:

$\underline{q}_{\pm 0}$ —+ $\underline{q}_{+1}$ —+ $\underline{q}_{+2}$ … —+ $\underline{q}_{+k}$ —+ …
$\underline{q}_{\pm 0}$ —+ $\underline{q}_{-1}$ —+ $\underline{q}_{-2}$ … —+ $\underline{q}_{-k}$ —+ …

Thus, the symmetry of $_{\mathbf{z}}\underline{\mathbf{Q}}$ implied by the isomorphism of $_{\mathbf{w}}\underline{\mathbf{Q}}$ and $_{\mathbf{w*}}\underline{\mathbf{Q}}$, means there is no longer a total ordering (unless the "definiteness is reversed" in $_{\mathbf{w*}}\underline{\mathbf{Q}}$), so <u>*we do **not** have*</u>:

$$\ldots \;\to\; \underline{\mathbf{q}}_{-k} \;\to\; \ldots \;\to\; \underline{\mathbf{q}}_{-2} \;\to\; \underline{\mathbf{q}}_{-1} \;\to\; \underline{\mathbf{q}}_{\pm 0} \;\to\; \underline{\mathbf{q}}_{+1} \;\to\; \underline{\mathbf{q}}_{+2} \ldots \;\to\; \underline{\mathbf{q}}_{+k} \;\to\; \ldots$$

***Defining* <u>C</u>um *Addition*, <u>C</u>um +**

Is that it? Does that define our space of **<u>C</u>um**s and their inverses, **&** its **<u>q</u>**ualifiers space? In a word, "No". As yet, we have not defined **<u>C</u>um +**, the addition of **<u>C</u>um**s, in this expanded space of **<u>C</u>um**s, not to mention their **<u>C</u>um ×** inverses (**<u>C</u>um +** must be defined in order to define "**+**" in **<u>q</u>**ualifiers space). But in **W-<u>C</u>um**, this addition is defined as: $\underline{\mathbf{C}}_k + \underline{\mathbf{C}}_n = \underline{\mathbf{C}}_{\max\{k,n\}}$. Correspondingly, in **W*-<u>C</u>um**, this addition would be defined under the corresponding isomorphic image rules: $\underline{\mathbf{C}}_{-k} + \underline{\mathbf{C}}_{-n} = \underline{\mathbf{C}}_{\min\{-k,-n\}}$. But how should **<u>C</u>um +** be defined for a "mixed **<u>C</u>um**", $\underline{\mathbf{C}}_{-k} + \underline{\mathbf{C}}_n$, i.e., with subscripts opposite in sign?

Originally (in ***<u>E</u>.<u>D</u>. <u>Brief</u> #5***), **<u>C</u>um +** was defined as the union set: $\underline{\mathbf{C}}_k + \underline{\mathbf{C}}_n := \underline{\mathbf{C}}_k \cup \underline{\mathbf{C}}_n$. But, in our previous case, $\mathbf{k < n}$ meant that $\underline{\mathbf{C}}_k \subset \underline{\mathbf{C}}_n$, i.e., the set-number $\underline{\mathbf{C}}_k$ was entirely contained within set-number $\underline{\mathbf{C}}_n$. Nevertheless, we shall define the "mixed sum" as: $\underline{\mathbf{C}}_{-k} + \underline{\mathbf{C}}_n := \underline{\mathbf{C}}_{-k} \cup \underline{\mathbf{C}}_n$. And, as before, we can invoke a notion of "subtraction" (indicated by a tilde: **~**) via a notion of "set difference",

$$\underline{\mathbf{C}}_m \sim \underline{\mathbf{C}}_{-k} := \underline{\mathbf{C}}_n \quad\Leftrightarrow\quad \underline{\mathbf{C}}_m = \underline{\mathbf{C}}_n + \underline{\mathbf{C}}_{-k}$$

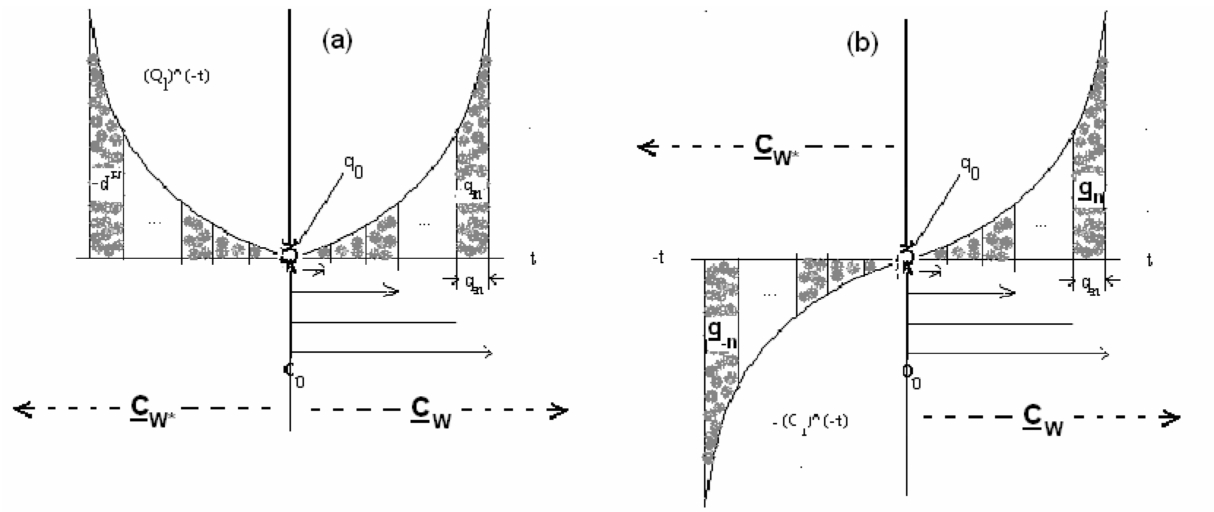
Letting $\mathbf{m = 0}$ and $\mathbf{k = n}$, we have a statement relative to $\mathbf{C}_{\pm 0}$, the "null-**<u>C</u>um**":

If we do have $\mathbf{C}_{\pm 0} = \underline{\mathbf{C}}_n + \underline{\mathbf{C}}_{-n}$, then $\mathbf{C}_{\pm 0} \sim \underline{\mathbf{C}}_{-n} := \underline{\mathbf{C}}_n$.

And since $\mathbf{C}_{\pm 0}$ is "like **±0** additively", $\mathbf{C}_{\pm 0} \sim \underline{\mathbf{C}}_{-n} = \sim\underline{\mathbf{C}}_{-n} = \underline{\mathbf{C}}_n$, i.e., we might be led to think that "the opposite (**~**) of $\underline{\mathbf{C}}_{-n}$ is like $\underline{\mathbf{C}}_n$", or conversely, that "the opposite of $\underline{\mathbf{C}}_n$ is like $\underline{\mathbf{C}}_{-n}$".

<u>Figure 1</u> attempts to illustrate two options of how "negative **<u>C</u>um**s" may exist. Generally, sets $\underline{\mathbf{C}}_n$ **&** $\underline{\mathbf{C}}_{-k}$ appear disjoint except for having the $\mathbf{C}_{\pm 0}$ element in common (since $\mathbf{C}_{\pm 0} \subset \underline{\mathbf{C}}_z$, for all **z** in **Z**). So, subtracting one set from the other, say $\underline{\mathbf{C}}_n \sim \underline{\mathbf{C}}_{-k}$, or "netting out" the $\underline{\mathbf{C}}_{-k}$ elements in $\underline{\mathbf{C}}_n$ (only $\mathbf{C}_{\pm 0}$ elements), yields all of the $\underline{\mathbf{C}}_n$ set-numbers except $\mathbf{C}_{\pm 0}$, so $\underline{\mathbf{C}}_n \sim \underline{\mathbf{C}}_{-k} = \underline{\mathbf{C}}_n \sim \mathbf{C}_{\pm 0} \approx \underline{\mathbf{C}}_n$, with the "≈" sign indicating "*'perhaps having' the same quality as*". Here, our "reasoning via set-analogy" says (when $\mathbf{-k = n}$) that set $\underline{\mathbf{C}}_{-n}$ is *like* (≈) $\underline{\mathbf{C}}_n$, and we are led to see $\underline{\mathbf{C}}_{-n}$ as much *like* $\underline{\mathbf{C}}_n$: $\underline{\mathbf{C}}_{-n} \approx \underline{\mathbf{C}}_n$. But, is this "alleged likeness" as **1** is to **1** ("exact equality"), or is it "likeness" as $|\mathbf{-1}|$ is to $|\mathbf{+1}|$ (opposite, but "equal in some qualitative sense")? In later sections, we shall explore our options more precisely.

<u>Figure 1</u>: Illustration of Possible Nature of **Z-<u>C</u>um** (**<u>C</u>z**) Space



Key Question: *"Is '$\underline{\mathbf{C}}_{+1} + (\underline{\mathbf{C}}_{+1})^{-1} = \mathbf{C}_{\pm 0}$', or not?"*

Again, we do not know what the precise relationship is without an assumption, a postulate perhaps. As of yet, *we do not know if* $\underline{\mathbf{C}}_{+1} + (\underline{\mathbf{C}}_{+1})^{-1} = \mathbf{C}_{\pm 0}$ *or not.* If so, then by applying our linear qualo-operator, $\underline{\partial}$, to both sides of our qualitative equality, we would have that --

$$\underline{\partial}(\underline{\mathbf{C}}_{+n} + \underline{\mathbf{C}}_{-n}) = \underline{\partial}\underline{\mathbf{C}}_{+n} + \underline{\partial}\underline{\mathbf{C}}_{-n} = \underline{\partial}\mathbf{C}_{\pm 0} \Rightarrow \underline{\mathbf{q}}_{+n} + \underline{\mathbf{q}}_{-n} = \mathbf{q}_{\pm 0} \text{ (by definition, } \underline{\mathbf{q}}_k := \underline{\partial}\underline{\mathbf{C}}_k).$$

So *(if* $\underline{\mathbf{C}}_{+n} + \underline{\mathbf{C}}_{-n} = \mathbf{C}_{\pm 0}$, *then)* $+\underline{\mathbf{q}}_{-n} = -(\underline{\mathbf{q}}_{+n}) = -\underline{\mathbf{q}}_{+n}$, *would be the additive inverse of* $\underline{\mathbf{q}}_{+n}$.

(As in ***Brief #5***, $\underline{\partial}$ defines "**Z**-**q**ualifier addition": $\underline{\partial}(\underline{\mathbf{C}}_{z1} + \underline{\mathbf{C}}_{z2}) = \underline{\partial}\underline{\mathbf{C}}_{z1}$ "+" $\underline{\partial}\underline{\mathbf{C}}_{z1} = \underline{\mathbf{q}}_{z1} + \underline{\mathbf{q}}_{z2}$.)

We can use these results to interpret the suitability of Figures **1(a)** or **1(b)** to represent an illustrative model of $\underline{\mathbf{C}}_{+n}$ and $\underline{\mathbf{C}}_{-n}$. In Figure **1(a)**, we have equal but opposite "qualitative areas" representing $\underline{\mathbf{C}}_{+n}$ (positive area) and $\underline{\mathbf{C}}_{-n}$ (negative area), yet their differentials or **q**ualifiers, $\underline{\mathbf{q}}_{-n}$ and $\underline{\mathbf{q}}_{+n}$ (opposite areas) are pointing in the same direction (when we might prefer them to be opposite since $+\underline{\mathbf{q}}_{-n} = -\underline{\mathbf{q}}_{+n}$). In Figure **1(b)**, we have equal positive "qualitative areas" representing either $\underline{\mathbf{C}}_{+n}$ or $\underline{\mathbf{C}}_{-n}$, but their **q**ualifiers (also equal positive areas) are pointing in the opposite direction (which we prefer). Thus, neither figure is the "perfect model" for what might be illustrated graphically, so we'll let the reader choose which s/he prefers (if either) as a guide to their understanding.

<u>To summarize so far</u>: Motivated by our desire for a **Cum** space that contains **Cum** × inverses, we have constructed a "**Z**-**Cum** base space" under an extended **Cum** × and an extended **Cum** + (**Cum**-addition). The corresponding Integer-**Q**ualifier space, $_{\mathbf{Z}}\underline{\mathbf{Q}}$, is then defined as the set of "differentials of the **Cum**s", $\{\underline{\mathbf{q}}_k := \underline{\partial}\underline{\mathbf{C}}_z$ for all $\mathbf{z}$ of $\mathbf{Z}\}$, or as the union of the **q**ualifier spaces: $_{\mathbf{Z}}\underline{\mathbf{Q}} := {_{\mathbf{W}}\underline{\mathbf{Q}}} \cup {_{\mathbf{W}^*}\underline{\mathbf{Q}}}$. The addition of **q**ualifiers, '$\underline{\mathbf{q}}_{z1} + \underline{\mathbf{q}}_{z2}$' is defined by $\underline{\partial}$ acting on the defined **Cum** sum, '$\underline{\mathbf{C}}_{z1} + \underline{\mathbf{C}}_{z2}$'.

The resulting *versions* **I** *and* **II** have the same **Cum** × and **Cum** + operations in common, i.e., they *share the same originating "<u>Base space</u>"* := < {$\underline{\mathbf{C}}_z$}, **Cum** ×, **Cum** +, $\underline{\partial}$( ), $\underline{\int}$( ); id(×) = $\mathbf{C}_{\pm 0}$ = id(+) >, while their corresponding **q**ualifier spaces will share the same **q**ualifier "×" multiplication.

4. Version I: $\underline{\mathbf{C}}_{+1} + (\underline{\mathbf{C}}_{+1})^{-1} = \mathbf{C}_{\pm 0}$, and defining '**Z**-**Cum**' space and $_{\mathbf{Z}}\underline{\mathbf{Q}}$'s "×"

Version* I *postulate: $\underline{\mathbf{C}}_{+1} + (\underline{\mathbf{C}}_{+1})^{-1} = \mathbf{C}_{\pm 0}$

In Version **I(a)**, *the current* **F**.***E***.***D***. *version of* $_{\mathbf{Z}}\underline{\mathbf{Q}}$, we define $\underline{\mathbf{C}}_{+1} + (\underline{\mathbf{C}}_{+1})^{-1}$ as $\mathbf{C}_{\pm 0}$, using the *"Integers"* (**Z**) as our quantitative base set, along with this postulate:

<u>Additive Identity / Amalgamative Sum Postulate</u>: In **Z**-**Cum**, $\underline{\mathbf{C}}_{+1} + (\underline{\mathbf{C}}_{-1})^{-1} = \underline{\mathbf{C}}_{+1} + \underline{\mathbf{C}}_{-1} := \mathbf{C}_{\pm 0}$, i.e., a *qualitative equality exists* between the sum, $\underline{\mathbf{C}}_{+1} + \underline{\mathbf{C}}_{-1}$, and its **Cum** × identity element, $\mathbf{C}_{\pm 0}$, so that we have $\mathbf{C}_{\pm 0} = \underline{\mathbf{C}}_{+1} + \underline{\mathbf{C}}_{-1}$ as an ***<u>amalgamative</u>*** *sum*.

Therefore, the differential of this sum, $\underline{\partial}(\underline{\mathbf{C}}_{+1} + \underline{\mathbf{C}}_{-1}) = \underline{\partial}\underline{\mathbf{C}}_{+1} + \underline{\partial}\underline{\mathbf{C}}_{-1} = \underline{\mathbf{q}}_{+1} + \underline{\mathbf{q}}_{-1} = \mathbf{q}_{\pm 0} = \underline{\partial}\mathbf{C}_{\pm 0}$, says that $\underline{\mathbf{q}}_{+1} + \underline{\mathbf{q}}_{-1}$ *is an "amalgamative sum" equal (reducible) to* $\mathbf{q}_{\pm 0}$, the identity element of integer open **q**ualifier space, $\underline{\mathbf{OQ}}_{\mathbf{Z}}$, all possible finite sums of $_{\mathbf{Z}}\underline{\mathbf{Q}}$ elements. This, in turn, says that $+\underline{\mathbf{q}}_{-z} = -\underline{\mathbf{q}}_{+z}$ for all $\mathbf{z}$ in $\mathbf{Z}$. Since $+\underline{\mathbf{q}}_{-z}$ is the × inverse of $+\underline{\mathbf{q}}_{+z}$ and since $-\underline{\mathbf{q}}_{+z}$ is the + inverse of $+\underline{\mathbf{q}}_{-z}$, this equality means *the* ***<u>additive</u> inverse element*** *is also the* ***<u>multiplicative</u> inverse element*** *for any element* $\underline{\mathbf{q}}_z$ *of* $_{\mathbf{Z}}\underline{\mathbf{Q}}$ ***!***

Version* I *"cumulation formulas"

But under its defined **Cum** + addition, **Z**-**Cum** is no longer closed under that addition! Instead, those non-**Z**-**Cum** sums are simply non-amalgamative sums of **q**ualifiers! We shall now understand that this

$\underline{\text{C}}\text{um}+$ closure set, $<\mathbf{Z}\text{-}\underline{\text{C}}\text{um}, \underline{\text{C}}\text{um}+>$, is $\underline{\text{O}}$pen integer-$\underline{\text{Q}}$ualifier space under its "+" addition: $\underline{\text{OQ}}_{\text{Z}}$ $:= <{}_{\text{Z}}\underline{\text{Q}}$, "+" $>$. Thus, the immediate result of this closure is (for $\mathbf{n} > \mathbf{k}$ of $\mathbf{Z}$):

In general, for $\mathbf{k} < \mathbf{n}$ in $+\mathbf{W}$, we have --

$$\underline{\text{C}}_{+n} + \underline{\text{C}}_{-k} = \int(\partial\underline{\text{C}}_{+t})_{+t \text{ in } [\pm 0,\, n] \text{ of } w} + \int(\partial\underline{\text{C}}_{-u})_{-u \text{ in } [\pm 0,\, -k] \text{ of } -w}$$

$$= \sum(\underline{\text{q}}_{+t})_{+t \text{ in } [\pm 0,\, n]} + \sum(\underline{\text{q}}_{-u})_{-u \text{ in } [\pm 0,\, -k]}$$

$$= (\text{q}_{\pm 0} + \underline{\text{q}}_{+1} + \underline{\text{q}}_{+2} + \dots + \underline{\text{q}}_{+n}) + (\text{q}_{\pm 0} + \underline{\text{q}}_{-1} \dots + \underline{\text{q}}_{-k})$$

$$= (\text{q}_{\pm 0} + \text{q}_{\pm 0}) + (\underline{\text{q}}_{-1} + \underline{\text{q}}_{+1}) + \dots + (\underline{\text{q}}_{-k} + \underline{\text{q}}_{+k}) + (\underline{\text{q}}_{+k+1} + \dots + \underline{\text{q}}_{+n});$$ rearranging in pairs:

$$= (\text{q}_{\pm 0}) \quad + \quad (\text{q}_{\pm 0}) \quad + \dots + \quad (\text{q}_{\pm 0}) + (\underline{\text{q}}_{+k+1} + \dots + \underline{\text{q}}_{+n}),$$

$$= \underline{\text{q}}_{+k+1} + \underline{\text{q}}_{+k+2} + \dots + \underline{\text{q}}_{+n-1} + \underline{\text{q}}_{+n},$$ by definition of $\text{q}_{\pm 0} = \text{id}(+)$.

Then, for $\mathbf{k} = \mathbf{n}$, we have " the symmetric 'zero-sum' " = " a sum of 'zero pairs' ", as postulated --

$$\underline{\text{C}}_{+n} + \underline{\text{C}}_{-n} = (\underline{\text{q}}_{-1} + \underline{\text{q}}_{+1}) + (\underline{\text{q}}_{-2} + \underline{\text{q}}_{+2}) + \dots + (\underline{\text{q}}_{-n} + \underline{\text{q}}_{+n}) = (\text{q}_{\pm 0}) + (\text{q}_{\pm 0}) + \dots + (\text{q}_{\pm 0}) = \text{q}_{\pm 0}.$$

Similarly, for $-\mathbf{k} < -\mathbf{n}$ in $-\mathbf{W}$, we have

$$\underline{\text{C}}_{-n} + \underline{\text{C}}_{-k} = \underline{\text{q}}_{-n} + \underline{\text{q}}_{(-n)-1} + \underline{\text{q}}_{(-n)-2} + \dots + \underline{\text{q}}_{(-k)+1} + \underline{\text{q}}_{(-k)} = \underline{\text{q}}_{(-k)} + \underline{\text{q}}_{(-k)+1} + \dots + \underline{\text{q}}_{(-n)-2} + \underline{\text{q}}_{(-n)-1} + \underline{\text{q}}_{-n}.$$

***Defining* "×", *the* q*ualifier multiplication operation in the* $_{Z^*}\underline{Q}$ "q*ualifier base space*"**

The only remaining task for defining our "base space" of $\mathbf{Z}$-$\underline{\text{C}}$ums, and $_{Z^*}\underline{Q}$ $\mathbf{Z}$-$\underline{\text{q}}$ualifiers space, is to define the multiplication operation "×"on the $_{Z^*}\underline{Q}$ space. Recall in ***E.D.Brief #5*** on **N**atural $\underline{\text{q}}$ualifiers space, $_{N}\underline{Q}$, we listed four possible alternatives for axiomatic definition of its multiplication operation as:

1) $\underline{\text{q}}_k$ "×" $\underline{\text{q}}_n$ $:=$ $\underline{\text{q}}_{n+k}$, commutative [F.*E*.*D*. name: *"meta-heterosis convolute product"*];
2) $\underline{\text{q}}_k$ "×" $\underline{\text{q}}_n$ $:=$ $\underline{\text{q}}_k + \underline{\text{q}}_{n+k}$, non-commutative [F.*E*.*D*. name: *"meta-catalysis evolute product"*];
3) $\underline{\text{q}}_k$ "×" $\underline{\text{q}}_n$ $:=$ $\underline{\text{q}}_{n+k} + \underline{\text{q}}_n$, non-commutative [F.*E*.*D*. name: *"**double-**«**aufheben**» evolute product"*];
4) $\underline{\text{q}}_k$ "×" $\underline{\text{q}}_n$ $:=$ $\underline{\text{q}}_k + \underline{\text{q}}_{n+k} + \underline{\text{q}}_n$, commutative [F.*E*.*D*. name: *"meta-genealogical evolute product"*].

Definition **3** was selected for the **N**atural $\underline{\text{q}}$ualifiers, $_{N}\underline{Q}$, then for the **W**hole $\underline{\text{q}}$ualifiers, $_{W}\underline{Q}$, and then implicitly also for the $_{W^*}\underline{Q}$, as:

In $_{W}\underline{Q}$: $\underline{\text{q}}_{w1}$ "×" $\underline{\text{q}}_{w2}$ $:=$ $\underline{\text{q}}_{w1+w2} + \underline{\text{q}}_{w2}$, for $\underline{\text{q}}_{w1}$ and $\underline{\text{q}}_{w2}$ both in $_{W}\underline{Q}$;

In $_{W^*}\underline{Q}$: $\underline{\text{q}}_{-w1}$ "×" $\underline{\text{q}}_{-w2}$ $:=$ $\underline{\text{q}}_{-(w1+w2)} + \underline{\text{q}}_{-w2}$, for $\underline{\text{q}}_{-w1}$ and $\underline{\text{q}}_{-w2}$ both in $_{W^*}\underline{Q}$;

However, we still need to define "×" when one factor is in $_{W}\underline{Q}$ and the other factor is in $_{W^*}\underline{Q}$. To do this, we must keep in mind our *"new requirements"* for any such $\underline{\text{q}}$ualifier multiplication, "×", namely:

a) Under "×", $(\underline{\text{q}}_{+1})^{z}$ must generate the **z**th Cumulum: $(\underline{\text{q}}_{+1})^{+z} = \underline{\text{q}}_{+1} + \dots + \underline{\text{q}}_{+z}$, if $\mathbf{z} > \mathbf{0}$, or

$(\underline{\text{q}}_{-1})^{-z} = \underline{\text{q}}_{-1} + \dots + \underline{\text{q}}_{+z}$, if $\mathbf{z} < \mathbf{0}$.

b) "×" should be commutative, reflecting the "pure symmetry of the $\mathbf{Z}$-$\underline{\text{C}}$um and $_{Z}\underline{Q}$ spaces".

c) Under $\underline{\text{C}}$um ×, $\underline{\text{C}}_{+z} \times \underline{\text{C}}_{-z} = \text{C}_{\pm 0}$, so "×" may mirror this pattern on $_{Z}\underline{Q}$: $\underline{\text{q}}_{+z}$ "×" $\underline{\text{q}}_{-z}$ $:=$ $\text{q}_{\pm 0}$.

Only Definition **4** above fits these requirements, so it is selected as the "new, axiomatic, commutative multiplication operation definition" for all ${}_{\mathbf{Z}}\underline{\mathbf{Q}}$ elements:

$$\underline{\mathbf{q}}_{z1} \times \underline{\mathbf{q}}_{z2} \;:=\; \underline{\mathbf{q}}_{z1} + \underline{\mathbf{q}}_{z1+z2} + \underline{\mathbf{q}}_{z2}$$ *(the meta-genealogical evolute product* rule).

We note that "Requirement **c)**" is also satisfied:

$$\underline{\mathbf{q}}_{-n} \times \underline{\mathbf{q}}_{+n} = \underline{\mathbf{q}}_{-n} + \underline{\mathbf{q}}_{(-n)+(+n)} + \underline{\mathbf{q}}_{+n} = \underline{\mathbf{q}}_{-n} + \underline{\mathbf{q}}_{+n} + \underline{\mathbf{q}}_{-n+n} = (\underline{\mathbf{q}}_{-n} + \underline{\mathbf{q}}_{+n}) + \mathbf{q}_{\pm 0} = (\mathbf{q}_{\pm 0}) + \mathbf{q}_{\pm 0} = \mathbf{q}_{\pm 0}.$$

It is also a bit comforting that this new "$\times$" reduces to old $\times$ for ${}_{\mathbf{N}}\underline{\mathbf{Q}}$, e.g., when "squaring" a **q**ualifier:

$$(\underline{\mathbf{q}}_{n})^2 = \underline{\mathbf{q}}_{n} \times \underline{\mathbf{q}}_{n} = \underline{\mathbf{q}}_{n} + \underline{\mathbf{q}}_{n+n} + \underline{\mathbf{q}}_{n} = \underline{\mathbf{q}}_{n} + \underline{\mathbf{q}}_{n} + \underline{\mathbf{q}}_{n+n} = (\underline{\mathbf{q}}_{n} + \underline{\mathbf{q}}_{n}) + \underline{\mathbf{q}}_{2n} = \underline{\mathbf{q}}_{n} + \underline{\mathbf{q}}_{2n}.$$

F.*E*.*D*. postulates that the ${}_{\mathbf{Z}}\underline{\mathbf{Q}}$ product obeys *"the meta-genealogical evolute product-rule"* for good reason. The two factors interacting to produce a "product" can be regarded as two "parents" interacting to [re-]produce a "child", or, perhaps more appropriately, to [*re*]*produce*, or *form*, a "Family" := "Parent_**z1** + Child_**(z1+z2)** + Parent_**z2**":

"Parent_**z1**" *interacting with* "Parent_**z2**" [*re*-]*produces* " Parent_**z1** + Child_-**(z1+z2)** + Parent_**z2**"

$$\underline{\mathbf{q}}_{z1} \qquad \times \qquad \underline{\mathbf{q}}_{z2} \qquad := \qquad \underline{\mathbf{q}}_{z1} \quad + \quad \underline{\mathbf{q}}_{z1+z2} \quad + \quad \underline{\mathbf{q}}_{z2}$$

Thus, when stated in terms of "parents", "child", and "family", we can more readily understand the phrase: "*meta-genealogical evolut*[*e*]-tion", as meaning '*beyond* parents to family', and thus we can better appreciate the name: "*meta-genealogical evolute product*". To me, this *interpretation* within the **F.*E*.*D*.** model helps gives it *"life"* and *"history"* in a very essential and human way!

We now have "qualo-fractions" and "qualo-differences"!

Qualo-Fractions: With the existence of $\times$ inverses in ${}_{\mathbf{Z}}\underline{\mathbf{Q}}$, *qualitative fractions*, or *"qualo-fractions"*, $\underline{\mathbf{q}}_{z1}/\underline{\mathbf{q}}_{z2}$, emerge, as the product of a "qualo-numerator" ($\underline{\mathbf{q}}_{z1}$) with an $\times$ *inverse* as "qualo-denominator" $(\underline{\mathbf{q}}_{z2})^{-1}$ in $\underline{\mathbf{OQ}}_{\mathbf{Z}}$, Integer **O**pen **Q**ualifier Space:

$$\underline{\mathbf{q}}_{z1}/\underline{\mathbf{q}}_{z2} := \underline{\mathbf{q}}_{z1} \times (\underline{\mathbf{q}}_{z2})^{-1} := \underline{\mathbf{q}}_{z1} \times \underline{\mathbf{q}}_{(-z2)} = \underline{\mathbf{q}}_{z1} + \underline{\mathbf{q}}_{z1-z2} + \underline{\mathbf{q}}_{(-z2)} = \underline{\mathbf{q}}_{z1} + \underline{\mathbf{q}}_{z1-z2} - \underline{\mathbf{q}}_{(+z2)}.$$

Qualo-Differences: Via the $+$ inverses in ${}_{\mathbf{Z}}\underline{\mathbf{Q}}$, *qualitative differences*, or *"qualo-differences"*, $\underline{\mathbf{q}}_{z1} - \underline{\mathbf{q}}_{z2}$, arise: sums of "qualo-minuends" ($\underline{\mathbf{q}}_{z1}$) with $+$ *inverses* as "qualo-subtrahends" ($\underline{\mathbf{q}}_{z2}$), in $\underline{\mathbf{OQ}}_{\mathbf{Z}}$:

$$\underline{\mathbf{q}}_{z1} - \underline{\mathbf{q}}_{z2} := \underline{\mathbf{q}}_{z1} + (-\underline{\mathbf{q}}_{z2}) := \underline{\mathbf{q}}_{z1} + \underline{\mathbf{q}}_{(-z2)}.$$

Note that '$\underline{\mathbf{q}}_{z1}/\underline{\mathbf{q}}_{z2} \neq \underline{\mathbf{q}}_{z1} - \underline{\mathbf{q}}_{z2}$': Although in ${}_{\mathbf{Z}}\underline{\mathbf{Q}}$ we have that $(+\underline{\mathbf{q}}_{+z})^{-1} = +\underline{\mathbf{q}}_{(-z)} = -\underline{\mathbf{q}}_{+z}$, this does not imply that '$\mathbf{q}_{z1}/\mathbf{q}_{z2}$' is qualitatively equal to '$\mathbf{q}_{z1} - \mathbf{q}_{z2}$'. Why? As the equalities above show, they are not: $\underline{\mathbf{q}}_{z1}/\underline{\mathbf{q}}_{z2} := \underline{\mathbf{q}}_{z1} + \underline{\mathbf{q}}_{z1-z2} - \underline{\mathbf{q}}_{z2} \neq \underline{\mathbf{q}}_{z1} - \underline{\mathbf{q}}_{z2}$. But why not? Because we cannot generally interchange a '$+$' operation with an '$\times$' operation in ${}_{\mathbf{Z}}\underline{\mathbf{Q}}$!

Note on "circular flow of signs": It is worth noticing that the use of exponent (superscript) and subscript notation results in a *circular flow of signs* ($-$ or $+$) around the $\underline{\mathbf{q}}$ symbol as center, that yields equivalences (for any $\underline{\mathbf{q}}_{z}$ in ${}_{\mathbf{Z}}\underline{\mathbf{Q}}$): $(-\underline{\mathbf{q}}_{+z})^{+1} = (+\underline{\mathbf{q}}_{-z})^{+1} = (+\underline{\mathbf{q}}_{+z})^{-1}$.

Can we solve '$\underline{\mathbf{AX}} = \underline{\mathbf{B}}$' or '$\underline{\mathbf{A}} + \underline{\mathbf{X}} = \underline{\mathbf{B}}$' in '$\underline{\mathbf{OQ}}_{\mathbf{Z}*}$' space?

In high school algebra, one repeatedly solves quantitative equations of the form $\mathbf{3x} = \mathbf{1}$ or $\mathbf{5} + \mathbf{x} = \mathbf{10}$, or generally: $\mathbf{ax} = \mathbf{b}$, and $\mathbf{a} + \mathbf{x} = \mathbf{b}$. In open **q**ualifier space, $\underline{\mathbf{OQ}}_{\mathbf{Z}}$, we might attempt a general solution to

$\underline{\mathbf{AX}} = \underline{\mathbf{B}}$, where $\underline{\mathbf{A}} = \sum \underline{\mathbf{q}}_{+t} \text{ over } \{a\}$ and $\underline{\mathbf{B}} = \sum \underline{\mathbf{q}}_{+t} \text{ over } \{b\}$ are sums in $\underline{\mathbf{OQ}}_{\mathbf{Z}}$. If $< \underline{\mathbf{OQ}}_{\mathbf{Z}}, \times >$ is a group, we can apply $\underline{\mathbf{A}}^{-1} = [\sum \underline{\mathbf{q}}_{+t} \text{ over } \{a\}]^{-1} = \sum \underline{\mathbf{q}}_{-t} \text{ over } \{a\}$ to both sides of the equation, then re-associate (as a group allows) to obtain: $(\underline{\mathbf{A}}^{-1} \times \underline{\mathbf{A}}) \times \underline{\mathbf{X}} = \underline{\mathbf{A}}^{-1} \times \underline{\mathbf{B}} \Rightarrow (\mathbf{q}_0) \times \underline{\mathbf{X}} = \underline{\mathbf{X}} = \underline{\mathbf{A}}^{-1} \times \underline{\mathbf{B}} = \underline{\mathbf{B}} \times \underline{\mathbf{A}}^{-1} = \underline{\mathbf{B}}/\underline{\mathbf{A}}$, and we would thus establish that *'qualitative fractions'*, or *'qualo-fractions'*, are the solutions in $\underline{\mathbf{OQ}}_{\mathbf{Z}}$.

Because $-\underline{\mathbf{A}}$ will exist, $\underline{\mathbf{A}} + \underline{\mathbf{X}} = \underline{\mathbf{B}}$ may be solvable as $(-\underline{\mathbf{A}} + \underline{\mathbf{A}}) + \underline{\mathbf{X}} = -\underline{\mathbf{A}} + \underline{\mathbf{B}} = \underline{\mathbf{B}} - \underline{\mathbf{A}}$, if **+associativity** holds in this Version **I** case, which would mean that such *'difference sums'*, or such *'qualo-differences'*, are the solutions in $\underline{\mathbf{OQ}}_{\mathbf{Z}}$.

***Do we want** '$\underline{\mathbf{q}}_z + \underline{\mathbf{q}}_z = \underline{\mathbf{q}}_z$', **or** 'associativity of +', **in** $\underline{\mathbf{OQ}}_{\mathbf{Z}}$?*

However, as presently defined, Version **I** sometimes lacks associativity of addition because of the "give-away idea" requirement which says that $\underline{\mathbf{q}}_z + \underline{\mathbf{q}}_z = \underline{\mathbf{q}}_z$ (or "$\underline{\mathbf{q}}_z - \underline{\mathbf{q}}_z = \underline{\mathbf{q}}_z$", i.e., give away idea $\underline{\mathbf{q}}_z$ and you still have it) for any **q**ualifier $\underline{\mathbf{q}}_z$ in ${}_{\mathbf{Z}}\underline{\mathbf{Q}}$. Yet, we have that $+\underline{\mathbf{q}}_{-z} = -\underline{\mathbf{q}}_{+z}$, so:

$$(\underline{\mathbf{q}}_{+z} + \underline{\mathbf{q}}_{+z}) + \underline{\mathbf{q}}_{-z} = (\underline{\mathbf{q}}_{+z}) - \underline{\mathbf{q}}_{+z} = \mathbf{q}_{\pm 0}, \text{ but}$$

$$\underline{\mathbf{q}}_{+z} + (\underline{\mathbf{q}}_{+z} + \underline{\mathbf{q}}_{-z}) = \underline{\mathbf{q}}_{+z} + (\mathbf{q}_{\pm 0}) = \underline{\mathbf{q}}_{+z}, \text{ for all } \mathbf{z} \text{ in } \mathbf{Z}.$$

Together, the result is *a contradiction -- unless* **+associativity** *is allowed not to hold for some cases* in for $\underline{\mathbf{OQ}}_{\mathbf{Z}}$. Therefore, we must choose between **a)** having **+associativity** in all cases, OR **b)** permitting non-associativity, but maintaining the "give-away idea" requirement ($\underline{\mathbf{A}} + \underline{\mathbf{A}} = \underline{\mathbf{A}}$). If we choose to keep $\underline{\mathbf{A}} + \underline{\mathbf{A}} = \underline{\mathbf{A}}$, we have Version **I** as established, accepting a degree of non-associativity in $\underline{\mathbf{OQ}}_{\mathbf{Z}}$. If, on the other hand, we require **+associativity**, we must "give up 'the give-away idea' idea," and presumably gain that $< \underline{\mathbf{OQ}}_{\mathbf{Z}}, + >$ and $< \underline{\mathbf{OQ}}_{\mathbf{Z}}, \times >$ are both commutative groups, having distributivity of $\times$ over $+$. Remarkably, that choice might suggest that $< \underline{\mathbf{OQ}}_{\mathbf{Z}}, +, \times; \mathbf{id}(+) = \mathbf{q}_{\pm 0} = \mathbf{id}(\times) >$ *would be a "super-field!"*-- a hitherto undefined concept in abstract algebra!

But alas, such enthusiasm is short-lived since, in abandoning $\underline{\mathbf{A}} + \underline{\mathbf{A}} = \underline{\mathbf{A}}$ for all $\underline{\mathbf{A}}$ in $\underline{\mathbf{OQ}}_{\mathbf{Z}}$, we no longer have the conditions that implied $\mathbf{id}(+) = \mathbf{q}_{\pm 0} = \mathbf{id}(\times)$, as proven in Appendix **A1** of ***Brief #6***. So, in adopting **+associativity &** abandoning $\underline{\mathbf{A}} + \underline{\mathbf{A}} = \underline{\mathbf{A}}$, we would lose $\mathbf{q}_{\pm 0} = \mathbf{id}(\times)$ since:

$$\underline{\mathbf{q}}_z \times \mathbf{q}_{\pm 0} = \underline{\mathbf{q}}_z + \underline{\mathbf{q}}_{z\pm 0} + \mathbf{q}_{\pm 0} = \underline{\mathbf{q}}_z + \underline{\mathbf{q}}_z \;\nparallel\; \underline{\mathbf{q}}_z!$$

Thus, we wouldn't even have a multiplicative identity element, let alone a *"super-field"*! So, motivation is missing to abandon $\underline{\mathbf{A}} + \underline{\mathbf{A}} = \underline{\mathbf{A}}$, less being gained than lost thereby. (Oh, but how exciting to imagine a *"super-field"* possibility!).

In '**C**': $(\mathbf{i})^{-1} = -\mathbf{i}$: In our Version **I** spaces, we have $(\underline{\mathbf{C}}_z)^{-1} = -\underline{\mathbf{C}}_z$ for all $\underline{\mathbf{C}}_z$ of $\underline{\mathbf{C}}_{\mathbf{Z}}$, and $(+\underline{\mathbf{q}}_{+z})^{-1} = -\underline{\mathbf{q}}_{+z}$ for all $\underline{\mathbf{q}}_z$ of ${}_{\mathbf{Z}}\underline{\mathbf{Q}}$. By way of contrast, the space of the **C**omplex numbers ("**C**") is the only well-known [qualo-]quantitative space which has elements such that $(\mathbf{x})^{-1} = -\mathbf{x}$ (true only for $\mathbf{x} = +\mathbf{i}$ and $\mathbf{x} = -\mathbf{i}$)! Only $+\mathbf{i}$ and $-\mathbf{i}$ in all of **C** have their ***multiplicative inverses*** the ***same as*** their ***additive inverses***.

We conclude this section with a *"real world" application* of our Version **I** ontological spaces. Let $\underline{\mathbf{q}}_{+m}$:= { ontology behind/of a "**m**atter particle" }. Then $(+\underline{\mathbf{q}}_{+m})^{-1} = +\underline{\mathbf{q}}_{-m} = -\underline{\mathbf{q}}_{+m}$ = { ontology "behind"/of an "**anti**-**m**atter particle" }. With matter and anti-matter "particles" modeled as "ontological inverses", we describe their behavior in "matter/anti-matter interactions" as "mutually-annihilatory":

$$\underline{\mathbf{q}}_{-m} \times \underline{\mathbf{q}}_{+m} = \mathbf{q}_{\pm 0} \quad \text{and} \quad \underline{\mathbf{q}}_{+m} + \underline{\mathbf{q}}_{-m} = \underline{\mathbf{q}}_{+m} - \underline{\mathbf{q}}_{+m} = \mathbf{q}_{\pm 0}.$$

Such behavior has, of course, been confirmed by countless experiments in "particle" physics. F.E.D.'s model result matches those observational results.

The above definitions and relationships complete our model of the Version **I** spaces: **Z**-**C̲um** and $_{\mathbf{Z}}\underline{\mathbf{Q}}$:

$< \underline{C}_Z := \{ \underline{C}_z \}, \underline{C}um \times, \underline{C}um +, \partial(\,); C_{\pm 0} = id(\times,+); \text{`}\underline{C}_{+1} + (\underline{C}_{+1})^{-1} = C_{\pm 0}\text{'}; {}_{Z}\underline{Q} := \{ \underline{q}_z \}, \times, +, \int(\,); q_{\pm 0} = id(\times,+) >$

In essence, Version **I**'s $_{Z}\underline{Q}$ (or $\underline{OQ}_Z$) is a space that includes "negative ideas", which not only can nullify (to $q_{\pm 0}$) a "natural idea" under ×, but which can also "negate" it completely (to $q_{\pm 0}$) under +! Thus, $\underline{q}_{(-n)}$ *ideas have an inescapable "minus-ness" about them!*

5. Version II: Z*-C̲um and $_{Z^*}\underline{Q}$ spaces -- $\underline{C}_1 + \underline{C}_1 \nsupseteq C_0$

Version* II *postulate: $\underline{C}_n + \underline{C}_{n^*} \nsupseteq C_0$

In Version **II**, we define an alternate **Z***-**C̲um** and $_{Z^*}\underline{Q}$ space, based upon a postulated *qualitative inequality* of C_0 and $\underline{C}_1 + (\underline{C}_1)^{-1}$, using the "**2-D W***hole Numbers"* as our quantitative base set (defined in Appendix **A0**), together with the following postulate:

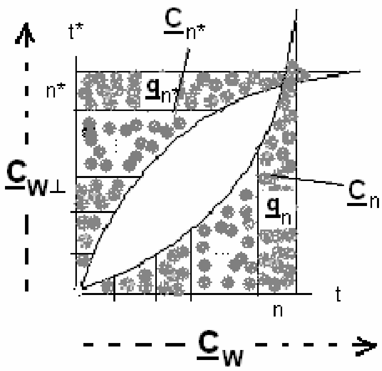
<u>*Non-Amalgamative Sum Postulate*</u>: In **Z***-**C̲um**, $\underline{C}_1 + (\underline{C}_1)^{-1} = \underline{C}_1 + \underline{C}_{1^*} \nsupseteq C_0$, i.e., a *qualitative inequality* exists between the sum, $\underline{C}_1 + (\underline{C}_1)^{-1}$, and C_0 and any other cumulation $\underline{C}_x$ in $\underline{C}_{Z^*}$ (implying that $\underline{q}_1 + \underline{q}_{1^*}$ is a *"**<u>non</u>**-amalgamative sum."*)

Let $(\underline{C}_1)^{-1} := \underline{C}_{1^*}$, where **+1*** is in **Z***. This postulate says that the sum, $\underline{C}_1 + \underline{C}_{1^*}$, cannot be reduced to any other element of $_{Z^*}\underline{Q}$. Thus, neither can the sum $\underline{C}_n + \underline{C}_{n^*}$. So, the differential of any such sum, $\underline{\partial}(\underline{C}_n + \underline{C}_{n^*}) = \underline{\partial}\underline{C}_n + \underline{\partial}\underline{C}_{n^*} = \underline{q}_n + \underline{q}_{n^*}$, *is considered to be a "**<u>non</u>**-amalgamative sum"* in our corresponding open **q̲**ualifier space, $\underline{OQ}_{Z^*}$.

These non-amalgamative sums still do not guarantee additive (+) **associativity** in Version **II**'s open **q̲**ualifier space, $\underline{OQ}_{Z^*}$ space, because when a + inverse *does* exist (shown below), those sums do amalgamate. Thus, we cannot ensure that the system $< \underline{OQ}_{Z^*}, +; q_0 = id(+) >$ is a commutative group (see Appendix **A3**) – indeed, it appears that it is not such a group. Such is also true for Version **I**'s system $< \underline{OQ}_Z, +; q_0 = id(+) >$, because its amalgamative sums (due to + inverses) fail to provide universal **+associativity** in its open **q̲**ualifier space, $\underline{OQ}_Z$ (see Appendix **A1**).

<u>**Figure 2**</u> is an attempt to illustrate two examples of what might occur in **Z***-**C̲um** space. Shown are possible "equal, but not opposite" qualitative areas of $\underline{C}_Z$ and $\underline{C}_{Z^*}$.

<u>**Figure 2**</u>: Illustration of Possible Nature of **Z***-**C̲um** ($\underline{C}_{Z^*}$) Space



Version **II** *"cumulation formulas"*

The **Cum** + **&** "+" definitions on $_{z*}\underline{\mathbf{Q}}$ elements [$\underline{\partial(\mathbf{C}_n + \mathbf{C}_{k*})}$:= $\underline{\partial \mathbf{C}_n}$ "+" $\underline{\partial \mathbf{C}_{k*}}$:= $\underline{\mathbf{q}}_n + \underline{\mathbf{q}}_{k*}$, where $\underline{\partial +}$:= "+"] then allow us to write the *basic cumulation formulas* for adding cumulations, $\underline{\mathbf{C}}_n$ and $\underline{\mathbf{C}}_{k*}$, and for their mixed-sum cumulation, $\underline{\mathbf{C}}_n + \underline{\mathbf{C}}_{k*}$.

In general, there is no need to compare $\mathbf{k < n}$ or $\mathbf{k > n}$ (since we are summing on different axes!):

$$\underline{\mathbf{C}}_n + \underline{\mathbf{C}}_{k*} = \int(\underline{\partial \mathbf{C}}_t)_{\mathbf{t} \text{ in } [0,\, n] \text{ of } \mathbf{w}} + \int(\underline{\partial \mathbf{C}}_{u*})_{\mathbf{u*} \text{ in } [\pm 0,\, k*] \text{ of } \mathbf{w}\perp}$$

$$= \sum(\underline{\mathbf{q}}_t)_{\mathbf{t} \text{ in } [0,\, n] \text{ of } \mathbf{w}} + \sum(\underline{\mathbf{q}}_{u*})_{\mathbf{u*} \text{ in } [\pm 0,\, k*] \text{ of } \mathbf{w}\perp}$$

$$= (\mathbf{q}_0 + \underline{\mathbf{q}}_1 + \underline{\mathbf{q}}_2 + \ldots + \underline{\mathbf{q}}_n) + (\underline{\mathbf{q}}_{k*} + \ldots + \underline{\mathbf{q}}_{1*} + \mathbf{q}_{0*})$$

$$= (\underline{\mathbf{q}}_{k*} + \ldots + \underline{\mathbf{q}}_{1*} + \mathbf{q}_{0*}) + (\mathbf{q}_0 + \underline{\mathbf{q}}_1 + \underline{\mathbf{q}}_2 + \ldots + \underline{\mathbf{q}}_n), \text{ by commutative rearranging}$$

$$= \underline{\mathbf{q}}_{k*} + \ldots + \underline{\mathbf{q}}_{1*} + \underline{\mathbf{q}}_1 + \underline{\mathbf{q}}_2 + \ldots + \underline{\mathbf{q}}_n, \text{ since } \mathbf{0^*} = \mathbf{0}, \text{ and } \mathbf{q}_0 = \mathbf{q}_{0*} = \mathbf{id(+)}.$$

Then for $\mathbf{k = n}$, we have the "symmetric sum" = "a sum of non-zero pairs"

$$\underline{\mathbf{C}}_n + \underline{\mathbf{C}}_{n*} = \underline{\mathbf{q}}_{k*} + \ldots + \underline{\mathbf{q}}_{1*} + \underline{\mathbf{q}}_1 + \underline{\mathbf{q}}_2 + \ldots + \underline{\mathbf{q}}_n$$

$$= (\underline{\mathbf{q}}_{1*} + \underline{\mathbf{q}}_1) + (\underline{\mathbf{q}}_{2*} + \underline{\mathbf{q}}_2) + \ldots + (\underline{\mathbf{q}}_{n*} + \underline{\mathbf{q}}_n), \text{ by commutative rearranging \& associating.}$$

In Version **II**, the same commutative *meta-genealogical evolute product* is employed on $\underline{\mathbf{C}}_{Z*}$'s **q**ualifier set, $_{z*}\underline{\mathbf{Q}}$. Its corresponding $\underline{\mathbf{OQ}}_{Z*}$ space may, in some cases, contain additive inverses, $-\underline{\mathbf{A}}$, for some sums $\underline{\mathbf{A}}$ (as exemplified immediately below), but $\underline{\mathbf{OQ}}_{Z*}$ does not necessarily have such "opposites".

We conclude with a *"real world" application* of our Version **II** ontological spaces. Let $\underline{\mathbf{q}}_n$:= **{** the ontology behind/of some "**n**ew particle" **}**. Then let $\underline{\mathbf{q}}_{n*}$:= **{** the ontology "behind"/of its "× inverse particle" **}**. Under Version **II**, we actually claim that its "× inverse particle"($\underline{\mathbf{q}}_{n*}$) cannot be its +opposite particle, since $\underline{\mathbf{q}}_n + \underline{\mathbf{q}}_{n*} \neq \mathbf{q}_0$. So, represented via qualitatively different ontologies for the "× inverse" ($\underline{\mathbf{q}}_{n*}$) and "+ inverse" ($-\underline{\mathbf{q}}_n$) **q**ualities, we might predict a new kind of behavior in/from an as yet undiscovered "**n**-particle" (from an "identity relation" established in Appendix **A0**):

$$\mathbf{q}_0 = \underline{\mathbf{q}}_n \times \underline{\mathbf{q}}_{n*} := \underline{\mathbf{q}}_n + \underline{\mathbf{q}}_{n*} + \underline{\mathbf{q}}_{n+n*}, \text{ therefore: } \underline{\mathbf{q}}_{n+n*} = -[\underline{\mathbf{q}}_n + \underline{\mathbf{q}}_{n*}].$$

or, in terms of $\mathbf{W}^{\mathbf{2\text{-}D}}$, the **2-D W***hole Number space* (see Appendix **A0**), we have --

$$\mathbf{q(0, 0)} = \underline{\mathbf{q}}\mathbf{(n, 0)} \times \underline{\mathbf{q}}\mathbf{(0, n)} := \underline{\mathbf{q}}\mathbf{(n, 0)} + \underline{\mathbf{q}}\mathbf{(0, n)} + \underline{\mathbf{q}}\mathbf{(n, n)},$$

$$\text{therefore: } \underline{\mathbf{q}}\mathbf{(n, n)} := -[\,\underline{\mathbf{q}}\mathbf{(n, 0)} + \underline{\mathbf{q}}\mathbf{(0, n)}\,].$$

Such an **n**-particle might be thought of as having its left-aspect, $\underline{\mathbf{q}}\mathbf{(n, 0)}$; its right-aspect, $\underline{\mathbf{q}}\mathbf{(0, n)}$; and its dual-aspect, or "full-aspect": $\underline{\mathbf{q}}\mathbf{(n, n)}$, which is the "opposite" (additive inverse) of the sum of its left- and right- aspects, which are × inverses of each other: $\underline{\mathbf{q}}\mathbf{(n, n)} := -[\,\underline{\mathbf{q}}\mathbf{(n, 0)} + \underline{\mathbf{q}}\mathbf{(0, n)}\,]$.

Appendix **A0** also defines a "dot product", "•", multiplication on $\mathbf{A} = \mathbf{(a, b)}$ & $\mathbf{B} = \mathbf{(c, d)}$ of $\mathbf{W}^{\mathbf{2\text{-}D}}$ as: $\mathbf{A \bullet B} = \mathbf{(ac, bd)} := \mathbf{ac + bd}$, and then shows that $\mathbf{A \bullet B = 0} \Leftrightarrow \mathbf{A \perp B} \Leftrightarrow \mathbf{A = (a, 0)}$ in

W & **B** = **(0, b)** in **W⊥** OR **A** = **(0, a)** in **W⊥** & **B** = **(b, 0)** in **W**. Plus, in **Z*** := < **W** ∪ **W⊥** > space, we can define a "dot product", "•", on $_{z*}$**Q** **q**ualifier elements as:

$$\underline{q}_A \bullet \underline{q}_B := \underline{q}_{A\bullet B} = \underline{q}_{ac+(bd)*} = \underline{q}_{ac} \times \underline{q}_{(bd)*} = \underline{q}_{ac} \times (\underline{q}_{bd})^* := \underline{q}_{ac}/\underline{q}_{bd}.$$

Since these "Version **II** results" originate from the orthogonal orientation of the **W⊥** space, via a flip across the **y = x** line, this might suggest an **n**-particle's predicted behavior. This "flip" may simply model a phenomenon such as polarized light, or an electron's 'half-spin" or "whole spin" state, in which case, a "particle/state" matching this arithmetical/algebraic model has already been discovered. Otherwise, the hypothesized **n**-particle behavior only points to a possible existence, which must, of course, be confirmed by empirical observation if it is to be deemed to be also an actual existence. *Version* **II***'s model merely expresses the possibility of such an* **n***-particle's existence with the behaviors indicated.*

The above definitions and relationships complete our definition of Version **II** spaces: **Z***-**C**um and $_{z*}$**Q**:

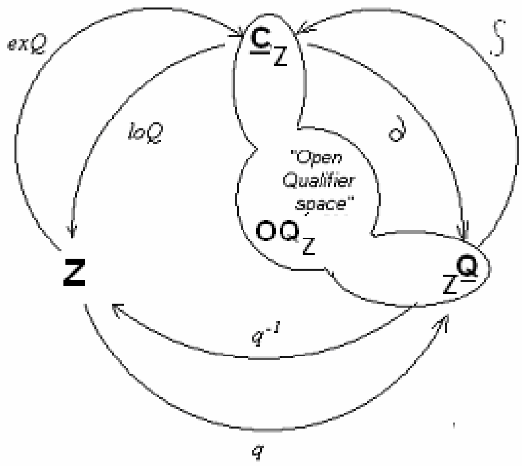
< $\underline{C}_{z*}$:= { $\underline{C}_{z*}$ }, **C**um ×, **C**um +, ∂( ); C_0 = id(×,+); '$\underline{C}_1$ + $(\underline{C}_1)^{-1}$ ⍭ C_0'; $_{z*}$**Q** := { $\underline{q}_{z+z*}$ }, ×, +, ∫( ); q_0 = id(×,+) >

In essence, Version **II**'s $_{z*}$**Q** (or **OQ**$_{z*}$) is a space permitting "orthogonal ideas" which can nullify (to q_0) a "natural idea" under ×, but does not usually "negate" it completely (to q_0) under +! Thus, $\underline{q}_{(n*)}$ *ideas have an ineluctable "orthogonal-flip" about them!*

6. Summary and Outlook

Figure 3 summarizes the functional relationships among **Z**, **C**$_z$, and $_z$**Q**, via inverses **q( )** and **q^{-1}( )**, **exQ( )** and **loQ( )**, and ∂**( )** and ∫**( )**. It also depicts "**O**pen **Q**ualifier Space" as containing both **C**$_z$ and $_z$**Q** spaces since "**OQ**$_z$ space" is the space of all possible finite sums (and products) which arise from $_z$**Q** qualifiers under addition and multiplication. [Here, "**Z**" refers to either the **Z** space or the **Z*** space.] We again note that although **OQ**$_z$ is operationally "closed" under × and + (i.e., contains all sums that its finite sums can generate as products or sums), **OQ**$_z$ is "open" in the sense of being "open to countless possible *'interpretations'* " of any sum or product in our modeling applications! Hence our term, **O***pen* **Z**-**Q***ualifier Space*(*s*). The algebraic natures of both the **OQ**$_z$ space and the **OQ**$_{z*}$ space have been delineated, in previous discussion herein, or in the appendices hereto. (Appendix **A3** summarizes these).

Figure 3: Relationships of **Z**, **C**$_z$, z**Q** with **q( )**, **q^{-1}( )**, **exQ( )**, **loQ( )**, ∂**( )**, ∫**( )**.



"Thought-full" Funful Note: In ***E*.*D***. Briefs **#5**, **#6**, and **#7**, we have come to entertain ourselves with the playful notion that '$\underline{\mathbf{OQ}}_{\mathbf{Z}}$ space' is quite "like a bunny rabbit's head", as is "artfully shown" in **Figure 3**. And now we "see" that between the bunny's ears are all **q**ualifiers $_{\mathbf{Z}}\underline{\mathbf{Q}}$ (at his right ear), and all **Cum**s $\underline{\mathbf{C}}_{\mathbf{Z}}$ (at his left ear), and many sums in between his ears. Could all these sums ('some' thoughts!) 'in between' represent the bunny's 'open-mind', thinking?! If so, then our bunny sure has taught us ('thought' us) a lot!

Our expanded systems of ontological **q**ualifier elements, $_{\mathbf{S}}\underline{\mathbf{Q}}$, ($\mathbf{S} = \mathbf{Z}$ or $\mathbf{Z^*}$), **&** binary operations $+$, $\times$ on $_{\mathbf{S}}\underline{\mathbf{Q}}$, *generally possess* **associativity** *in both* $+$ *and* $\times$, *and generally have* **distributivity** *of* $\times$ *over* $+$, *but not in all cases*. In the cases where **associativity** and **distributivity** do fail, the failure is due to the '$\underline{\mathbf{q}}_{\mathbf{n}} + \underline{\mathbf{q}}_{\mathbf{n}} = \underline{\mathbf{q}}_{\mathbf{n}}$' and '$\underline{\mathbf{q}}_{\mathbf{(-n)}} = -\underline{\mathbf{q}}_{\mathbf{n}}$' properties, creating a failure of $+$**associativity**, which may or may not produce failures in $\times$**associativity** and in $\times+$**distributivity**.

This algebraic system does, however, have the most unique property: '$\mathbf{id(+)} = \mathbf{q}_{\pm 0} = \mathbf{id(\times)}$', i.e., its 'Zero' of addition, and its 'One" of multiplication, are the same element! *This uniqueness was made possible by the* '$\underline{\mathbf{A}} + \underline{\mathbf{A}} = \underline{\mathbf{A}}$' *property of each element* $\underline{\mathbf{A}}$ *in the system – and ironically, it is this very* '$\underline{\mathbf{A}} + \underline{\mathbf{A}} = \underline{\mathbf{A}}$' *property which causes the failure of* $+$**associativity**!

Appendices **A1** and **A2** prove/disprove **associativity** (**A1**) and **distributivity** (**A2**) on both versions of Integer **O**pen **Q**ualifier Space: $\underline{\mathbf{OQ}}_{\mathbf{Z^*}}$ and $\underline{\mathbf{OQ}}_{\mathbf{Z}}$. Appendix **A3** shows that **O**pen Integer-**Q**ualifier space, $< \underline{\mathbf{OQ}}_{\mathbf{Z}}, \times >$ possibly, and $< \underline{\mathbf{OQ}}_{\mathbf{Z}}, + >$ are not commutative groups under their defined multiplication (even though they have inverses, $\times$ is non-associative because of $+$non-associativity in $\underline{\mathbf{OQ}}_{\mathbf{Z}}$). Appendix **A4** attempts to define "$\times\times$" and **^** on **Z**-**Cum** ($\underline{\mathbf{C}}_{\mathbf{Z}}$) as analogously as possible to the multiplication **&** exponentiation operations on **Z**, making $< \underline{\mathbf{C}}_{\mathbf{Z}}, \times, \times\times, \wedge >$ isomorphic to $< \mathbf{Z}, +, \times, \wedge >$. Using these results, speculation **&** proof are offered on *what* '$\underline{\mathbf{q}}_{\mathbf{k}} \wedge \underline{\mathbf{q}}_{\mathbf{n}}$' *might mean*, for **k**, **n** in **{–1**, **0**, **1}**.

The existence of **Z***-**Cum** ($\underline{\mathbf{C}}_{\mathbf{Z^*}}$) and $_{\mathbf{Z^*}}\underline{\mathbf{Q}}$ spaces, using $\mathbf{Z^*} =$ **2-D W***hole Numbers*, never permit any "epoch indices" $\tau' < \mathbf{0}$, when epochal time is $\tau \geq \mathbf{0}$. Thus, in Version **II**, one cannot ***mis***-interpret the "$\times$ inverse ontology" as a "going back in time" (but might interpret it as an 'orthogonal' "flip in time")!

The existence of the **Z**-**Cum** ($\underline{\mathbf{C}}_{\mathbf{Z}}$) and $_{\mathbf{Z}}\underline{\mathbf{Q}}$ spaces, however, allow for the "epoch index" $\tau' < \mathbf{0}$, when epochal time is $\tau \geq \mathbf{0}$. The "existence" of a **q**ualifier ontology, $\underline{\mathbf{q}}_{-\tau}$, $(-\tau < \mathbf{0})$ is but another "ontology" or "kind of being." As such, it exists in the mind -- so is possible, in that sense. *It does not imply the possibility of going backwards in time* (-- for that would require positing a "time-reverse ontology" for that purpose)!

This brief, in a way, represents a kind of "*finalé* for now-ee." Yet, there seems no end to the possibilities offered by **F**.***E***.***D***.*'s dialectical models of ontological space.*

-- *Joy-to-You* (July **2012**)

++ F.*E*.*D*. $\equiv$ **Foundation *Encyclopedia Dialectica***, authors of **A *Dialectical "Theory of Everything"* – *Meta-Genealogies of the Universe and of Its Sub-Universes*: A Graphical Manifesto**, **Volume 0**: **Foundations**. **www.dialectics.org** and/or **www.adventures-in-dialectics.org**

Appendix A0 -- Possible versions of Z^*-Cum and $_{Z^*}\underline{Q}$ spaces: $\underline{C}_{+1} + \underline{C}_{-1} \to C_0$

Let us define $S := Z^* := < W \cup W^*, + >$, all possible elements formed by unifying **W** with its isomorph **W*** under an extended addition operation. Then the basic problem that we attempting to solve is: "What is the possible (**Cum** ×)-inverse operation, $* := \wedge^{(-1)}$, which, acting twice on $\underline{C}_{z^*}$ space (and on $S := < W \cup W^* >$ by implication), leaves every element in those spaces unchanged." Thus, for any cum $\underline{C}_{z^*}$ for **z** in **S**, we have

$$[(\underline{C}_z)^{-1}]^{-1} = [(\underline{C}_z)^*]^* = [\underline{C}_{z^*}]^* = \underline{C}_{z^{**}} = \underline{C}_z \quad \text{and} \quad [(\underline{C}_S)^*]^* = [\underline{C}_{S^*}]^* = \underline{C}_{S^{**}} = \underline{C}_S,$$

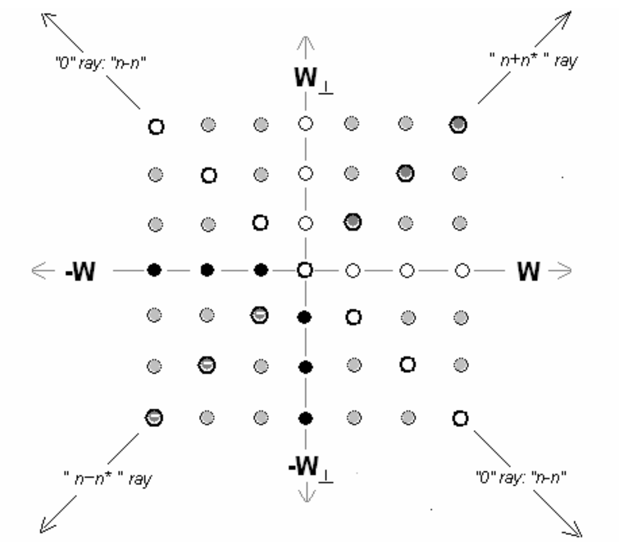
with the latter equation on the corresponding quantitative space **S** implying that $[S^*]^* = S^{**} = S$, for every **z** in $S = Z^*$. So, since the left equations say $Z^{**} = z$ for all **z** in $S = Z^*$, these equations in * imply:

$$(*)o(*) = (*)^2 = id(o), \text{ where "o" is the "composition of functions operation".}$$

C. Musès solved a very similar problem when he offered his "counter-complex" space; he simply stated the problem in terms of "**1**" (= **id(×)**, the × identity element in **R**eal space, treating * as a "number", say **ê** (:= *), in the resulting space. Thus, Musès solved "$\hat{e}\hat{e} = \hat{e}^2 = +1$" more generally than the **+1** and **–1** in **R**eal space. His solution set was $\{\hat{e}\} = \{+1, -1, +\varepsilon, -\varepsilon\} = K_4$, in which "**+1**" (the "multiplicative identity element" or "no action element"), "**–1**" (as the flip across $x = 0$, **y**-axis), '**+ε**' (the "flip" across the $y = x$ line), and '**–ε**' (the "flip" across the $y = -x$ line). Simply stated, K_4 under '**o**' is the **K**lein **4**-group. This general solution then shows us how to form possible configurations of Z^* space as outlined here (shown in **Figure A0-1**):

1) $\hat{e} = +1$, yields **W** space again, since $Z^* = < W \cup W^* > = < W > = W$.
2) $\hat{e} = -1$, yields the traditional integers, since $Z^* = < W \cup -W > = < Z > = Z$.
3) $\hat{e} = +\varepsilon$, suggests a new space where **W*** is orthogonal (perpendicular) to **W**, thus
   $Z^* := < W \cup W\perp > = < \{ (n, n^*) := n + n^*: n \text{ in } W, n^* \text{ in } W\perp \} >$.
4) $\hat{e} = -\varepsilon$, suggests another new space where **W*** is orthogonal (perpendicular) to **W**, thus
   $Z^* := < W \cup (-W\perp) > = < \{ (n, n^*) := n - n^*: n \text{ in } W, -n^* \text{ in } W\perp \} >$.

Figure A0-1: Illustration of Possible $Z^* = < W \cup W^* >$ Spaces



These four possible "types of **Z*** ", in turn, suggest four possible "types of **Z*-<u>C</u>um** space", $\underline{\mathbf{C}}_{\mathbf{Z^*}}$, with four possible orientations of cums in $\underline{\mathbf{C}}_{\mathbf{Z^*}} := \underline{\mathbf{C}}_{<\mathbf{W}\cup\mathbf{W^*}>}$. These orientations, shown in parts **(a)** and **(b)** of **<u>Figure A0-2</u>**, of **W*** and $\underline{\mathbf{C}}_{\mathbf{Z^*}}$ suggest the following cases (per the sequence listed for possible "**ê** = " above):

1) $\mathbf{Z^*} = <\mathbf{W}\cup\mathbf{W^*}, +> \quad = <\mathbf{W}> = \mathbf{W} \Rightarrow$ Quadrant **I(a)** (normal whole) locus of **<u>C</u>um** $\underline{\mathbf{C}}_{\mathbf{W^*}}$.
2) $\mathbf{Z^*} = <\mathbf{W}\cup(-\mathbf{W})>, + > = <\mathbf{Z}> \quad = \mathbf{Z} \Rightarrow$ Quadrant **II(a)** (normal integer) locus of **<u>C</u>um** $\underline{\mathbf{C}}_{\mathbf{W^*}}$.
3) $\mathbf{Z^*} = <\mathbf{W}\cup(\mathbf{W}_\perp), +> \quad = <\{(\mathbf{n}, \mathbf{m}) := \mathbf{n}+\mathbf{m^*}:\ \mathbf{n}$ in $\mathbf{W}$, $(+\mathbf{m})^*$ in $\mathbf{W}_\perp\}> \Rightarrow$ **QI(b)** locus of **<u>C</u>um** $\underline{\mathbf{C}}_{\mathbf{W^*}}$.
4) $\mathbf{Z^*} = <\mathbf{W}\cup(-\mathbf{W}_\perp), +> \quad = <\{(\mathbf{n}, \mathbf{m}) := \mathbf{n}-\mathbf{m^*}:\ \mathbf{n}$ in $\mathbf{W}$, $(-\mathbf{m})^*$ in $\mathbf{W}_\perp\}> \Rightarrow$ **QIII(b)** locus of **<u>C</u>um** $\underline{\mathbf{C}}_{\mathbf{W^*}}$.

We note that in Cases **1** and **3** the extended addition in **Z*** is first defined as by a vector sum, $(\mathbf{n}, \mathbf{m^*}) := \mathbf{n}+\mathbf{m^*}$, and that in Cases **2** and **4** the extended addition in **Z*** is first defined as by a vector difference, $(\mathbf{n}, \mathbf{m^*}) := \mathbf{n}-\mathbf{m^*}$. Note that when $\mathbf{W^*} = -\mathbf{W}$, the flip across the **y**-axis automatically implies that the sum $\mathbf{n}+\mathbf{n^*} = \mathbf{n}+(-\mathbf{n}) = \mathbf{0}$ appears only as the origin $(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. However, $\mathbf{W^*} = (-\mathbf{W}_\perp)$, the flip across the $\mathbf{y} = \mathbf{x}$ line, allows that the sum $\mathbf{n}+\mathbf{n^*} = (\mathbf{n}, \mathbf{n})$ appear not only as the origin, but as any number of points on either "**0 ray**" shown in **<u>Figure A0-1</u>**. This result allows for a "different kind of **Z*-<u>C</u>um** space" than the **Z-<u>C</u>um** we first sketched. Case **3**, Quadrant **I(b)**, also allows for the "$\mathbf{n}+\mathbf{n^*}$" ray to be defined; this case is selected to be our new **Z*** space. Thus, what we stated as our suspicion in the body of this article is confirmed in several ways.

I. **<u>Figure A0-2(a)</u>** (Case **2**) implies $\underline{\mathbf{C}}_1 + (\underline{\mathbf{C}}_1)^{-1} = \mathbf{C}_0$ & $\underline{\mathbf{C}}_{-1} = -\underline{\mathbf{C}}_1$, or "opposite-in-**<u>q</u>**uality" **<u>C</u>um**s, and
II. **<u>Figure A0-2(b)</u>** (Cases **3 & 4**) implies $\underline{\mathbf{C}}_1 + (\underline{\mathbf{C}}_1)^{-1} \not\ddagger \mathbf{C}_0$ & $\underline{\mathbf{C}}_1 + \underline{\mathbf{C}}_{1^*} \not\ddagger \mathbf{C}_0$, or "similar-in-**<u>q</u>**uality" **<u>C</u>um**s.

Letting $(\underline{\mathbf{C}}_1)^{-1} := (\underline{\mathbf{C}}_1)^* := \underline{\mathbf{C}}_{1^*}$:

Version **I**: $\underline{\mathbf{C}}_{+1} + \underline{\mathbf{C}}_{+1^*} = \mathbf{C}_{\pm 0}$, the sum of $\underline{\mathbf{C}}_{+1}$ and $\underline{\mathbf{C}}_{-1}$ reduces to the "amalgamative sum" $\mathbf{C}_{\pm 0}$.

Version **II**: $\underline{\mathbf{C}}_1 + \underline{\mathbf{C}}_{1^*} \not\ddagger \mathbf{C}_0$, the sum of $\underline{\mathbf{C}}_1$ and $\underline{\mathbf{C}}_{-1}$ is the "non-amalgamative sum" $\underline{\mathbf{C}}_1 + \underline{\mathbf{C}}_{1^*}$.

<u>Figure A0-2</u>: Illustration of Possible Nature(s) of the **Z-<u>C</u>um** ($\underline{\mathbf{C}}_{\mathbf{Z}}$) Space(s): $\underline{\mathbf{C}}_z + \underline{\mathbf{C}}_{z^*} =$ vs. $\not\ddagger \mathbf{C}_0$.

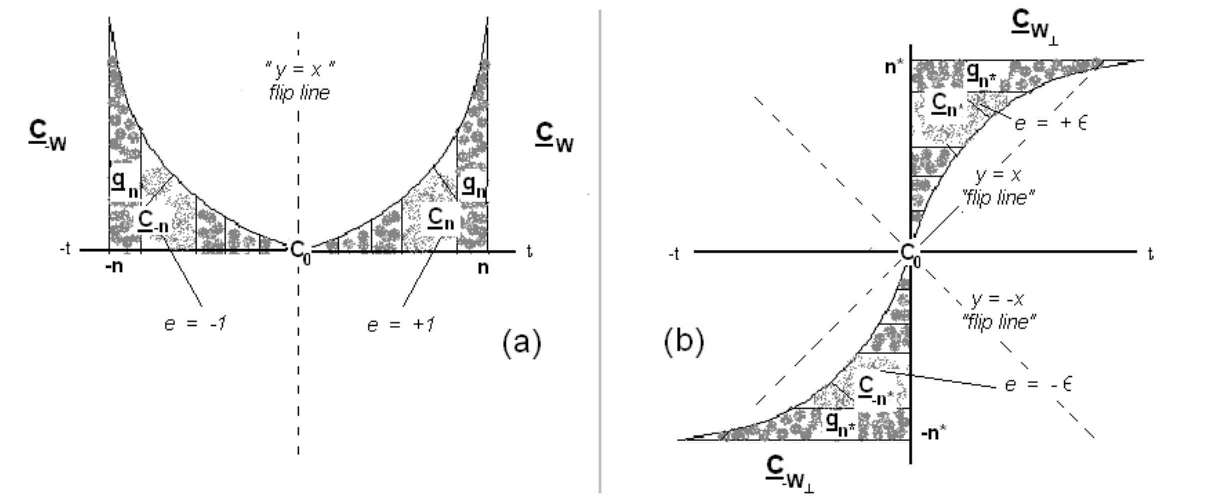
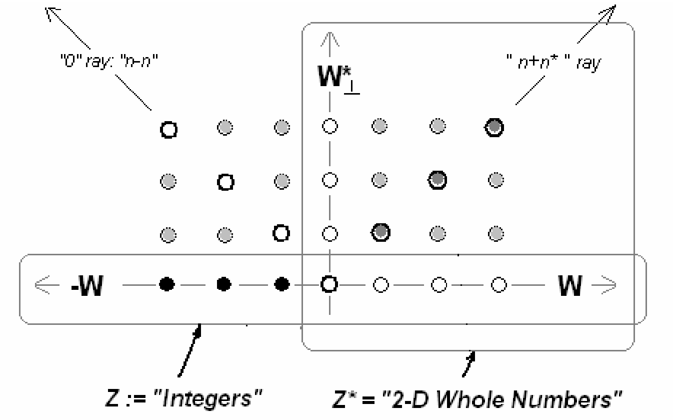


Figure A0-3: Illustration of Assumptions behind our **Integer Spaces**, **Z**, & **Z*** := **2-D Whole Numbers**.



For our Version **II**, we shall consider only Case **3** above, $\mathbf{Z^* := W^{2\text{-}D} := \langle W \cup W_\perp, + \rangle}$, in which we assume that the sum $\mathbf{n + n^* := 0}$ in some sense, i.e., **(n, n)** is viewed as a "difference" (+ as "±", perhaps) equivalent to **0** (see **Figure A0-3** & **Figure A0-4**). This enables us to claim that $\underline{\mathbf{q}}_\mathbf{n}$'s "×inverse", $\mathbf{(\underline{q}_n)^* = \underline{q}_{n^*}}$, is not its "+inverse", or "opposite", $\underline{\mathbf{q}}_{\mathbf{n^*}} \neq -\underline{\mathbf{q}}_\mathbf{n}$, or $\underline{\mathbf{q}}_{\mathbf{n^*}} \neq -\underline{\mathbf{q}}_\mathbf{n}$. So, modeled as ontologies for "×inverse" ($\underline{\mathbf{q}}_{\mathbf{n^*}}$) & "+inverse" ($-\underline{\mathbf{q}}_\mathbf{n}$) **q**ualities, we have a general identity, which to me has the form of a "conservation law" in physics:

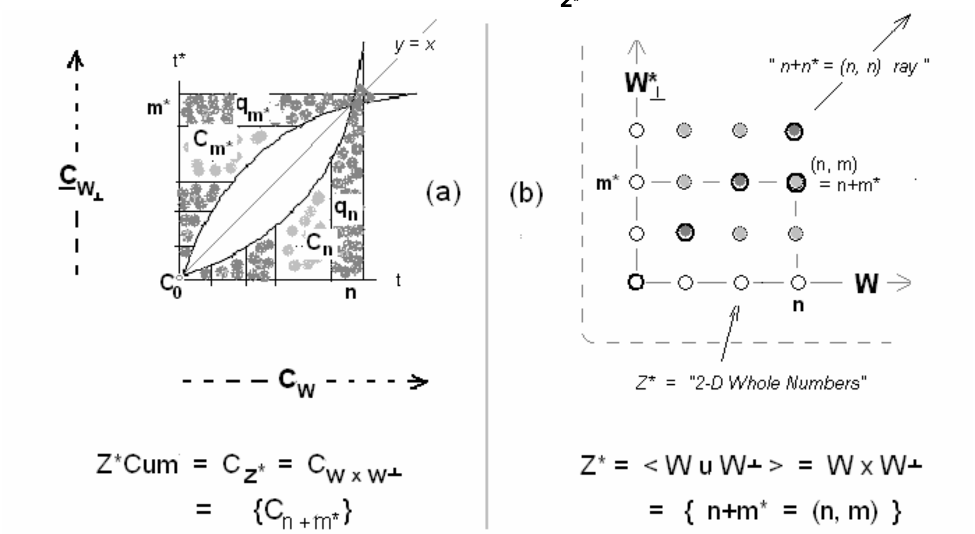
$$\mathbf{q_0} \;=\; \underline{\mathbf{q}}_\mathbf{n} \times \underline{\mathbf{q}}_{\mathbf{n^*}} \;:=\; \mathbf{q_n} + \mathbf{q_{n^*}} + \mathbf{q_{n+n^*}} \;=\; \mathbf{q_0}, \text{ therefore: } \underline{\mathbf{q}}_{\mathbf{n+n^*}} \;=\; -[\,\underline{\mathbf{q}}_\mathbf{n} + \underline{\mathbf{q}}_{\mathbf{n^*}}\,].$$

Or, in terms of $\mathbf{W^{2\text{-}D}}$, the **2-D W***hole Number Space*, we have:

$$\mathbf{q(0, 0) \;=\; \underline{q}(n, 0) \times \underline{q}(0, n) \;:=\; \underline{q}(n, 0) + \underline{q}(0, n) + \underline{q}(n, n)}, \text{ therefore: } \mathbf{\underline{q}(n, n) \;:=\; -[\underline{q}(n, 0) + \underline{q}(0, n)]}.$$

So, our proposed space might have its addition as: $\mathbf{a \pm b^* := (a, b) = (a, 0)}$ "±" $\mathbf{(0, b)}$, where each is thought of as representing its left-aspect, **(a, 0)**, its right-aspect, **(0, b)**, or its dual-aspect, or "full-aspect", **(a, b)**. If $\mathbf{b = a}$, then $\mathbf{(a, a^*) = a + a^* := 0}$, with the left-aspect and right aspect being × inverses of each other. In $\underline{\mathbf{OQ}}_{\mathbf{Z^*}}$, this would imply the existence of an "opposite" (additive inverse) of the sum of its left and right aspects, $-[\,\underline{\mathbf{q}}_\mathbf{n} + \underline{\mathbf{q}}_{\mathbf{n^*}}\,]$, that is $\underline{\mathbf{q}}_{\mathbf{n+n^*}}$ (as shown above) -- though the $_{\mathbf{z^*}}\underline{\mathbf{Q}}$ **q**ualifier space does not necessarily/generally contain such "opposites."

Figure A0-4: Correspondence between **Z***-**Cum** ($\underline{\mathbf{C}}_{\mathbf{Z^*}}$) and **Z*** := **2-D Whole Numbers**.



We may also define a "dot product •" multiplication on $\mathbf{A = (a, b)}$ and $\mathbf{B = (c, d)}$ of $\mathbf{W^{2\text{-}D}}$ as:

$$\mathbf{A \bullet B \;=\; (ac, bd) \;:=\; ac + (bd)^*},$$

and then show that --

$$A\bullet B = 0 \Leftrightarrow A \perp B \Leftrightarrow A = (a, 0) \text{ in } W, \ \& \ B = (0, b) \text{ in } W_\perp, \text{ OR } A = (0, a) \text{ in } W_\perp \ \& \ B = (b, 0) \text{ in } W.$$

Furthermore, in $Z^* := \langle W \cup W_\perp \rangle$ space, meaning that $id(\times)_W = (1, 0)$ and $id(\times)_{W\perp} = (0, 1)$. So we can define a dot, "•", multiplication (an appropriate name on Z^* element "dots"?) on ${}_{Z^*}\underline{Q}$ qualifier elements, $id(\bullet)|_{Z^*} = (1, 1)$, where $A = (a, b)$, and $B = (c, d)$:

$$\underline{q}_A \bullet \underline{q}_B := \underline{q}_{A\bullet B} = \underline{q}_{ac+(bd)^*} = \underline{q}_{ac} \times \underline{q}_{(bd)^*} = \underline{q}_{ac} \times (\underline{q}_{bd})^* := \underline{q}_{ac}/\underline{q}_{bd}.$$

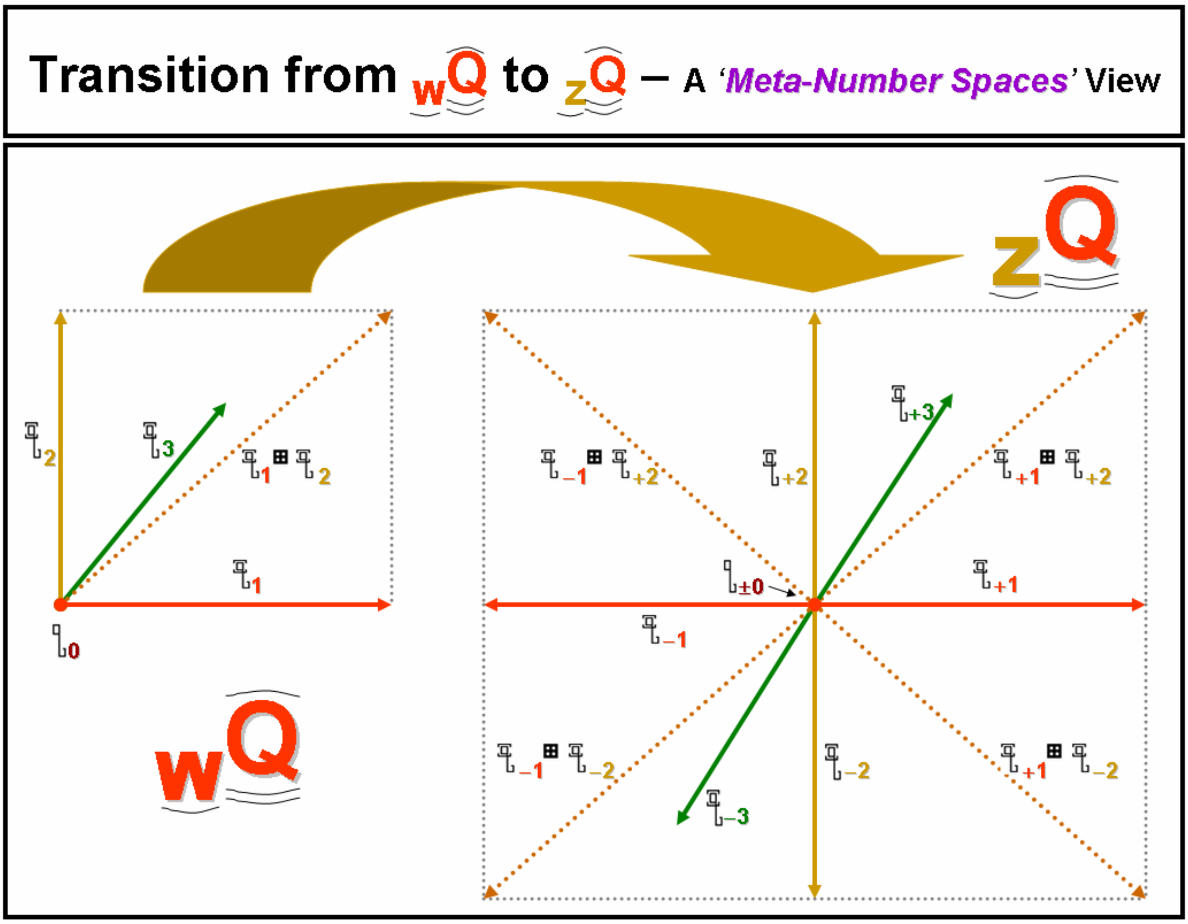
The "$y = x$ line" or the "$(n, n) = n + n^*$ dots" (in $W^{2\text{-}D}$) is a " line of 'self-inverses' ": $(n, n)^* = (n, n)$.
Only a defined vector addition has been necessary for our discussion, but for "completeness", we define another closed multiplication on Z^* as:

$$(a, b) \times (c, d) := (ac + bd, ad + bc).$$

[*Those familiar with Dr. Musès' "epsilon numbers" will note them implied here, as* $(a, b) = a + b\varepsilon$; $(c, d) = c + d\varepsilon$].

Our $Z^* := W^{2\text{-}D}$ is only a proposed space to offer as an alternative to Z and the additive inverses implied in ${}_Z\underline{Q}$. **Figure A0-5** is an **F.*E*.*D***. depiction of the transition from **W**hole **Q**ualifier space (${}_W\underline{Q}$) to Version **I**'s Integer **Q**ualifier space (${}_Z\underline{Q}$). Note the "**180°** opposite vectors" among all orthogonal elements of ${}_Z\underline{Q}$. *We now ask:* "Would the $(\underline{q}_n)^* = \underline{q}_{n^*}$ vectors appear as "**90°** non-opposite vectors" for a similar depiction of Version **II**'s **2-D W**hole Numbers **Q**ualifier space (${}_{Z^*}\underline{Q}$)?"

Figure A0-5: **F.*E*.*D***. Depiction of the Transition from the ${}_W\underline{Q}$ to the ${}_Z\underline{Q}$ qualifiers space.



Appendix A1 – Associativity of (×, +) in Open Qualifier Spaces?

Please interpret the findings below with the following note in mind.

> *Note*: Before determining **associativity** status, we note that in ordinary arithmetic, **1 / 2 + 3 × 4 – 3 / 6**, taken as **(1/2) + (2 × 5) – (3/6) = 10**, does not equal the former, taken, instead, as **1/(2 + 2) × (5 – 3)/6 = 1/12**. This "ambiguity" is resolved by the ***convention*** that multiplication and division operations are to be given priority-of-performance vis-à-vis addition and subtraction operations. In the case of the sometimes-***non***-**associativity** of $_{z}\underline{Q}$ addition, this might be resolved by adopting a somewhat similar ***convention***, e.g., by giving $_{z}\underline{Q}$ addition priority-of-performance over $_{z}\underline{Q}$ subtraction, after converting all $\underline{z} + \underline{q}_{-z}$ "additions" to $\underline{z} - \underline{q}_{+z}$ subtractions.

+ is not associative in $\underline{OQ}_z$ --

The "amalgamative" sums of $\underline{OQ}_z$ provide cases where **+associativity** fails:

$(\underline{q}_{+z} + \underline{q}_{+z}) + \underline{q}_{-z} \;=\; (\underline{q}_{+z}) - \underline{q}_{+z} \;=\; q_{\pm 0}$, but

$\underline{q}_{+z} + (\underline{q}_{+z} + \underline{q}_{-z}) \;=\; \underline{q}_{+z} + (q_{\pm 0}) \;=\; \underline{q}_{+z}$, thus + is not always associative in $\underline{OQ}_z$.

+ is not associative in $\underline{OQ}_{z*}$ --

The "non-amalgamative" sums of $\underline{OQ}_{z*}$ guarantee that the sum is the same no matter how it is grouped ("associated"), except in cases where a +inverse exists in $\underline{OQ}_{z*}$, as was shown for $\underline{q}_{n+n*} = -[\,\underline{q}_n + \underline{q}_{n*}\,]$. In such a case, we would have an example (as the one above, by replacing $\underline{q}_z$ with $[\,\underline{q}_{n+n*}\,]$) of the form:

$([\,\underline{q}_{n+n*}\,] + [\,\underline{q}_{n+n*}\,]) + -[\,\underline{q}_{n+n*}\,] \;=\; ([\,\underline{q}_{n+n*}\,]) - [\,\underline{q}_{n+n*}\,] \;=\; q_0$, but

$[\,\underline{q}_{n+n*}\,] + ([\,\underline{q}_{n+n*}\,] + -[\,\underline{q}_{n+n*}\,]) \;=\; [\,\underline{q}_{n+n*}\,] + (q_0) \;=\; [\,\underline{q}_{n+n*}\,]$, thus + is not always **associative** in $\underline{OQ}_{z*}$.

× is (or 'is not') associative in $\underline{OQ}_z$ --

We show that any triple product appears **associative**, but that such **×associativity** might depend on **+associativity** since every product is a sum.

$$\begin{aligned}
(\underline{q}_a \times \underline{q}_b) \times \underline{q}_c &= (\underline{q}_a + \underline{q}_{a+b} + \underline{q}_b) \times \underline{q}_c = (\underline{q}_a \times \underline{q}_c) + (\underline{q}_{a+b} \times \underline{q}_c) + (\underline{q}_b \times \underline{q}_c) \\
&= (\underline{q}_a + \underline{q}_{a+c} + \underline{q}_c) + (\underline{q}_{a+b} + \underline{q}_{a+b+c} + \underline{q}_c) + (\underline{q}_b + \underline{q}_{b+c} + \underline{q}_c) \\
&= (\underline{q}_a + \underline{q}_b + \underline{q}_c) + (\underline{q}_{a+b} + \underline{q}_{a+c} + \underline{q}_{b+c}) + (\underline{q}_{a+b+c}) \\
&= \underline{q}_a + \underline{q}_b + \underline{q}_c + \underline{q}_{a+b} + \underline{q}_{a+c} + \underline{q}_{b+c} + \underline{q}_{a+b+c} \\
\underline{q}_a \times (\underline{q}_b \times \underline{q}_c) &= \underline{q}_a \times (\underline{q}_b + \underline{q}_{b+c} + \underline{q}_c) = (\underline{q}_a \times \underline{q}_b) + (\underline{q}_a \times \underline{q}_{b+c}) + (\underline{q}_a \times \underline{q}_c) \\
&= (\underline{q}_a + \underline{q}_{a+b} + \underline{q}_b) + (\underline{q}_a + \underline{q}_{a+b+c} + \underline{q}_{b+c}) + (\underline{q}_a + \underline{q}_{a+c} + \underline{q}_c) \\
&= (\underline{q}_a + \underline{q}_b + \underline{q}_c) + (\underline{q}_{a+b} + \underline{q}_{a+c} + \underline{q}_{b+c}) + (\underline{q}_{a+b+c}) \\
&= \underline{q}_a + \underline{q}_b + \underline{q}_c + \underline{q}_{a+b} + \underline{q}_{a+c} + \underline{q}_{b+c} + \underline{q}_{a+b+c}
\end{aligned}$$

Thus, $(\underline{q}_a \times \underline{q}_b) \times \underline{q}_c$ *appears equal to* $\underline{q}_a \times (\underline{q}_b \times \underline{q}_c)$ before any sums are "amalgamated". Once they are amalgamated, depending upon the sum, the results may or may not be the same -- since + is not always **associative** in $\underline{OQ}_z$ *!*

× is (or 'is not') associative in $\underline{OQ}_{z*}$ --

The × in $\underline{OQ}_{z*}$ is the same as in $\underline{OQ}_z$, so the triple product-sums are identical. Thus, $(\underline{q}_a \times \underline{q}_b) \times \underline{q}_c$ *appears equal to* $\underline{q}_a \times (\underline{q}_b \times \underline{q}_c)$ before any sums are "amalgamated". Once they are amalgamated, depending upon the sum, the results may or may not be the same -- since + is not always **associative** in $\underline{OQ}_{z*}$ either!

Are + or × associative in $\underline{OQ}_W$***? Answer*****: + *is*, × *is not*.**

We now contrast the above findings with those for $\underline{OQ}_W$, **O**pen **W**hole-Numbers **Q**ualifier Space. First, we learn that **+associativity** may be *a bit* in question here also, since:

$(\underline{q}_n + \underline{q}_n) + \underline{q}_k \;=\; \underline{q}_n + \underline{q}_k$, but $\underline{q}_n + (\underline{q}_n + \underline{q}_k) \;=\; \underline{q}_n + \underline{q}_n + \underline{q}_k$, due to '$\underline{q}_n + \underline{q}_k$' being non-amalgamative.

But, doesn't this sum, by definition of sum as "ultimate attainable sum" beg that we apply whatever technique will further "reduce" it? This would mean that the first associated sum on the left, $(\underline{q}_n + \underline{q}_n) + \underline{q}_k$, defines the sum of three terms, which is '$\underline{q}_n + \underline{q}_n + \underline{q}_k$':

$$\underline{q}_n + \underline{q}_n + \underline{q}_k := (\underline{q}_n + \underline{q}_n) + \underline{q}_k := (\underline{q}_n + \underline{q}_n) + \underline{q}_k = \underline{q}_n + \underline{q}_k$$

Since we claimed that $\underline{q}_n + (\underline{q}_n + \underline{q}_k) = \underline{q}_n + \underline{q}_n + \underline{q}_k$, we have by transitivity, that

$(\underline{q}_n + \underline{q}_n) + \underline{q}_k = \underline{q}_n + (\underline{q}_n + \underline{q}_k) = \underline{q}_n + (\underline{q}_n + \underline{q}_k)$.

So, we shall claim **+associativity** in $< \underline{OQ}_W, + >$ – but we had to "argue our case" to claim it!

It is clear, however, that triple products, by *the double «**aufheben**» evolute product* rule for "×" in $\underline{OQ}_W$, are not **associative**, as these product-sums differ qualitatively, e.g., by the term $\underline{q}_{a+c}$ below: × is ***non***-associative in $< \underline{OQ}_W, \times >$:

$(\underline{q}_a \times \underline{q}_b) \times \underline{q}_c = (\underline{q}_b + \underline{q}_{a+b}) \times \underline{q}_c = (\underline{q}_b \times \underline{q}_c) + (\underline{q}_{a+b} \times \underline{q}_c) = (\underline{q}_c + \underline{q}_{b+c}) + (\underline{q}_c + \underline{q}_{a+b+c}) = \underline{q}_c + \underline{q}_{b+c} + \underline{q}_{a+b+c}$;

$\underline{q}_a \times (\underline{q}_b \times \underline{q}_c) = \underline{q}_a \times (\underline{q}_c + \underline{q}_{b+c}) = (\underline{q}_a \times \underline{q}_c) + (\underline{q}_a \times \underline{q}_{b+c}) = (\underline{q}_c + \underline{q}_{a+c}) + (\underline{q}_{b+c} + \underline{q}_{a+b+c}) = \underline{q}_c + \underline{q}_{a+c} + \underline{q}_{b+c} + \underline{q}_{a+b+c}$.

<u>Appendix A2</u> – Distributivity (of × over +) in <u>O</u>pen <u>Q</u>ualifier Spaces?

× ***does*** (***or*** '***does not***') ***distribute over + in*** $\underline{OQ}_Z$ ***and*** $\underline{OQ}_{Z^*}$ –

We may have a similar problem with **distributivity**, but the following shows that **distributivity** appears to hold except when the sums are amalgamated. Let $\underline{A} = \sum \underline{q}_{t \text{ over } \{a\}}$, $\underline{B} = \sum \underline{q}_{t \text{ over } \{b\}}$, and $\underline{C} = \sum \underline{q}_{t \text{ over } \{c\}}$. Then --

$$[\underline{A} + \underline{B}] \times \underline{C} = [\sum \underline{q}_{t\{a\}} + \sum \underline{q}_{t\{b\}}] \times (\sum \underline{q}_{t\{c\}}) = [\sum q_{t\{a\}}] \times (\sum \underline{q}_{t\{c\}}) + [\sum q_{t\{b\}}] \times (\sum q_{t\{c\}})$$

$$= (\sum \underline{q}_t \times \underline{q}_{u,\, t \text{ in } \{a\},\, u \text{ in } \{c\}}) + (\sum \underline{q}_k \times \underline{q}_{u,\, k \text{ in } \{a\},\, u \text{ in } \{c\}});$$

$$\underline{AC} + \underline{BC} = (\sum \underline{q}_{t\{a\}}) \times (\sum \underline{q}_{t\{c\}}) + (\sum \underline{q}_{t\{b\}}) \times (\sum \underline{q}_{t\{c\}})$$

$$= (\sum \underline{q}_t \times \underline{q}_{u,\, t \text{ in } \{a\},\, u \text{ in } \{c\}}) + (\sum \underline{q}_k \times \underline{q}_{u,\, k \text{ in } \{a\},\, u \text{ in } \{c\}}).$$

In both $\underline{OQ}_Z$ and $\underline{OQ}_{Z^*}$, the sums are identical, yet once those (sub-)sums which can be amalgamated, are amalgamated, we have the same difficulty with **+associativity** as before. Thus, we have **×+distributivity** only if the sums associate equally, i.e, produce the same end sum.

× ***does*** (***or*** '***does not***') ***distribute over non-amalgamative sums in*** $\underline{OQ}_W$ –

Finally, we show that × distributes over the non-amalgamative sum in $< \underline{OQ}_W, +, \times >$, from both sides, although the results are different (as we would expect since this "×" is non-commutative):

$$(\underline{q}_a + \underline{q}_b) \times \underline{q}_c = (\underline{q}_a \times \underline{q}_c) + (\underline{q}_b \times \underline{q}_c) = (\underline{q}_c + \underline{q}_{a+c}) + (\underline{q}_c + \underline{q}_{b+c}) = \underline{q}_c + \underline{q}_{a+c} + \underline{q}_{b+c};$$

$$\underline{q}_c \times (\underline{q}_a + \underline{q}_b) = (\underline{q}_c \times \underline{q}_a) + (\underline{q}_c \times \underline{q}_b) = (\underline{q}_a + \underline{q}_{a+c}) + (\underline{q}_b + \underline{q}_{b+c}) = \underline{q}_a + \underline{q}_b + \underline{q}_{a+c} + \underline{q}_{b+c}.$$

<u>Appendix A3</u> – Z or Z* <u>O</u>pen <u>Q</u>ualifier Spaces as Algebraic Systems

Despite the problems with **+associativity**, we may summarize our findings for each version's system or subsystem.

$< \underline{OQ}_Z, \times >$, $< \underline{OQ}_Z, \times >$ *are commutative 'almost-Groups'!* –

These subsystems each have a commutative, generally **associative** ×, **&** have all other Group properties: × closure, **id(×)**, and a $\mathbf{A}^{-1}$ for each **A**. Each subsystem has an × ***<u>non</u>*-associativity** only when its **+associativity** fails in its product-sums, as already explained (both above and below).

$< \underline{OQ}_Z, + >$, $< \underline{OQ}_{Z^*}, + >$ *are commutative 'almost-Groups'!* –

These subsystems each have a commutative, but non-associative +, but have all other Group properties: + closure, **id(+)**, and a **–A** for each **A**. In each, its **+associativity** fails due to $q_z + q_z = q_z$ and $q_{-z} = -q_z$.

For S = Z or Z*, *the systems* $< \underline{OQ}_S, +, \times;$ id(+) = q_0 = id(×) $>$ *are "far from" being 'super-fields'* –

Our math system of ontological <u>q</u>ualifier elements, ${}_S\underline{Q}$, and binary operations +, × on ${}_S\underline{Q}$, *generally exhibits* **associativity** *in both + and ×, and generally exhibits* **distributivity** *of × over +, <u>but **not** in all cases</u>*. In cases where these properties do fail, the failure is due to the '$\underline{q}_z + \underline{q}_z = \underline{q}_z$' and '$+\underline{q}_{-z} = -\underline{q}_{+z}$' properties creating a failure of **+associativity**, which may or may not produce failures also in **×associativity** and in **×+distributivity**.

Because the system is not always **associative** in either + or ×, nor always **distributive** for × over +, the entire system falls quite short of being an algebraic Field, or of being a Group in its subsystems.

This algebraic system does, however, have a most unique property: '**id(+) = q_0 = id(×)**', i.e., its 'Zero' of addition, and its 'One" of multiplication, are the same! *This uniqueness was made possible by the* '$\underline{A} + \underline{A} = \underline{A}$' *property of each element* $\underline{A}$ *in the system – and ironically, it is this* '$\underline{A} + \underline{A} = \underline{A}$' *property which causes failure of* **+associativity**, and hence of **×associativity**, and in **distributivity** of × over +!

< $\underline{OQ}_W$, +, ×; id(+) = q_0 = id(×) > *is a "distributive" system with* +associativity!

Our discussions above for $\underline{OQ}_W$ show that **<** $\underline{OQ}_W$, +, × **>** is a "**distributive**" system with **+associativity**, so that the subsystem **<** $\underline{OQ}_W$, +; **id(+) = q_0 >** is a **commutative monoid** (**:=** a **semigroup** with **id(+)**), as stated in ***Brief #6***.

Appendix A4 – Speculation / 'proof' on '$\underline{q}_k$^$\underline{q}_n$' for k, n in { –1, ±0, +1 }

Defining "××" *and* "^" *on* $\underline{C}_Z$ *and* $_Z\underline{Q}$ –

In **Appendix A3** of ***E.D. Brief #4***, "××" and "^" were defined on $_N\underline{Q}$ elements so that "××" and "^" were analogous to the multiplication and exponentiation in **N**. Similarly, in this appendix, we attempt to define an "××" on $\underline{C}_Z$ that is somewhat analogous to multiplication on **Z** so that < $\underline{C}_Z$, ×, ×× > is isomorphic to < **Z**, +, × >.

Since $<\underline{C}_{+n}\times>$ **m** times implies $\underline{C}_{+n}\times\underline{C}_{+n}\times\ldots\times\underline{C}_{+n} = [\underline{C}_{+n}]^{+m} := [(\underline{C}_{+1})^{+n}]^{+m}$. In order to make this result analogous to × in **W**, we define $\underline{C}_{+n} \times\times \underline{C}_{+m} := [(\underline{C}_{+1})^{+n}]^{+m} = (\underline{C}_{+1})^{+nm} = \underline{C}_{+nm} = \underline{C}_{+mn} = \underline{C}_{+m} \times\times \underline{C}_{+n}$, which says: **exQ(n) ×× exQ(m) = exQ(n × m)**, and that $\underline{C}_w \times\times C_0 = C_{0w} = C_0 = C_{w0} = C_0 \times\times \underline{C}_w$ for any **+w** of **+W**. Then for any **–n**, **–m** of **–W**, we shall regard the second factor $\underline{C}_{-m}$ as the "container of the number of times" that $<\underline{C}_{+n}\times>$ is to be performed, namely "$|-m|$ times".

Thus, we define $\underline{C}_{-n} \times\times \underline{C}_{-m} := [(\underline{C}_{+1})^{-n}]^{|-m|} = (\underline{C}_{+1})^{-n\times|-m|} = \underline{C}_{-n\times|-m|} = \underline{C}_{-m\times|-n|} = \underline{C}_{-m} \times\times \underline{C}_{-n}$. In essence, then, < (–**W**), × > ≈ < (+**W**), × > means the × in (–**W**) acts as: **(–1) × (–1) = (–1)**, as **(+1) × (+1) = (+1)** acts in **W**, and so does the corresponding **$\underline{C}$um** × in $\underline{C}_{-W}$ and $\underline{C}_{+W}$. Also, $id(\times)|_{(-W)} = (-1)$ as $id(\times)|_{(+W)} = (+1)$.

Similarly for **n, m > 0,** $\underline{C}_{-n} \times\times \underline{C}_{+m} := [(C_{+1})^{-n}]^{|m|} = (\underline{C}_{+1})^{-n|m|} = \underline{C}_{-nm} \neq \underline{C}_{+mn} = \underline{C}_{+m\times|-n|} = \underline{C}_{+m} \times\times \underline{C}_{-n}$ and for **n, m > 0**, $\underline{C}_{+m} \times\times \underline{C}_{-n} := [(\underline{C}_{+1})^{+m}]^{|-n|} = (\underline{C}_{+1})^{+nm} = \underline{C}_{+nm} = \underline{C}_{+mn} = \underline{C}_{+m} \times\times \underline{C}_{+n}$. Thus, we have defined a multiplication, ××, which is analogous to × on (+**W**) × (+**W**) and on (–**W**) × (+**W**), *but **not** analogous to* × on (–**W**) × (–**W**) nor to × on (+**W**) × (–**W**), as detaled below by "subregion". This is simply because the second factor is used to register an "absolute count" of repeated multiplication of the first factor. So, $C_{\pm0} \times\times \underline{C}_{-n} := [(\underline{C}_{+1})^{\pm0}]^{|-n|} = C_{\pm0}$, and, $\underline{C}_{-n} \times\times C_{\pm0} := [(\underline{C}_{+1})^{-n}]^{|0|} = C_{\pm0}$. Under ××, $C_{\pm0}$ serves as "annihilator", always reducing the product to itself. Therefore, a complete definition of ××: $\underline{C}_Z \times \underline{C}_Z \mapsto \underline{C}_Z$ defines ×× on each "quadrant"/"subregion":

- For $(\underline{C}_{+n}, \underline{C}_{+m})$ of quadrant $\underline{C}_{+W}\times\underline{C}_{+W}$: $\underline{C}_{+n} \times\times \underline{C}_{+m} := \underline{C}_{+nm}$; analogous to × on (+**W**) × (+**W**);
- For $(\underline{C}_{+n}, \underline{C}_{-m})$ of quadrant $\underline{C}_{+W}\times\underline{C}_{-W}$: $\underline{C}_{+n} \times\times \underline{C}_{-m} := \underline{C}_{+nm}$; *not analogous* to × on (+**W**) × (–**W**);
- For $(\underline{C}_{-n}, \underline{C}_{+m})$ of quadrant $\underline{C}_{-W}\times\underline{C}_{+W}$: $\underline{C}_{-n} \times\times \underline{C}_{+m} := \underline{C}_{-nm}$; analogous to × on (–**W**) × (+**W**);
- For $(\underline{C}_{-n}, \underline{C}_{-m})$ of quadrant $\underline{C}_{-W}\times\underline{C}_{-W}$: $\underline{C}_{-n} \times\times \underline{C}_{-m} := \underline{C}_{-nm}$; *not analogous* to × on (–**W**) × (–**W**);
- For $(C_{\pm0}, \underline{C}_{-m})$ of region $\{C_{\pm0}\}\times\underline{C}_Z$: $C_{\pm0} \times\times \underline{C}_{+k} := C_{\pm0}$; analogous to ±0 × on { ±0 } × **Z**;
- For $(\underline{C}_{+k}, C_{\pm0})$ of region $\underline{C}_Z \times \{C_{\pm0}\}$: $\underline{C}_{+k} \times\times C_{\pm0} := C_{\pm0}$; analogous to × ±0 on **Z** × { ±0 }.

The above results imply, *unlike* **+1** *and* **–1**, that $\underline{C}_{+1}$ is the right-identity for ×× on $\underline{C}_{+W}$ and $\underline{C}_{-1}$ is the right-identity for ×× on $\underline{C}_{-W}$: $\underline{C}_{+1} = id(\times\times|_{+W})$ and $\underline{C}_{-1} = id(\times\times|_{-W})$. This result probably follows from $C_{\pm0} = id(\times)$ on all of $\underline{C}_Z = \underline{C}_{W\cup(-W)}$, since $C_{\pm0} = id(+)$ also, $C_{\pm0}$ is both like **±0** and like **+1**; thus, we speculate that $\underline{C}_{-1}$ in $\underline{C}_{-W}$ is analogous to $\underline{C}_{+1}$ in $\underline{C}_{+W}$.

We extend the definition of ×× to **^** by: $\underline{C}_n$^$\underline{C}_m$:= [($\underline{C}_{+1}$)^n]^[$\underline{C}_m$] := [($\underline{C}_{+1}$)^n]$^{\wedge m}$:= [($\underline{C}_{+1}$)$^{n\wedge m}$] := $\underline{C}_{n\wedge m}$, which says: **exQ(n)^exQ(m) = exQ(n^m)**, & $\underline{C}_n$^$C_{\pm 0}$ = $\underline{C}_{n\wedge\pm 0}$ = $\underline{C}_{+1}$ = $C_{\pm 0}$^$\underline{C}_n$ ∀**n** in **Z**, except **n = ±0**. For **n = ±0 = u**, we note that $\text{Lim}_{u\to 0+}$**{ u^u } = +1**, so we define $C_{\pm 0}$^$C_{\pm 0}$:= $\underline{C}_1$ if $C_{\pm 0}$ is neared from above **±0**, & $C_{\pm 0}$^$C_{\pm 0}$:= $\underline{C}_{-1}$ if $C_{\pm 0}$ is neared from below **±0**: $\text{Lim}_{u\to 0+}$**{** $\underline{C}_u$^$\underline{C}_u$ **}** = $\underline{C}_{+1}$ & $\text{Lim}_{u\to 0-}$**{** $\underline{C}_u$^$\underline{C}_u$ **}** = $\underline{C}_{-1}$.** For the case of **n = ±0 & m = –1**, **±0^(–1) :=** "undefined", as **±0** has no ×inverse in **Z**, or in any purely-quantitative "**R**eal" space. However, since $C_{\pm 0}$ is its own ×inverse for the × of $\underline{C}_{z}$ (& $q_{\pm 0}$ is its own ×inverse for the × of $_{z}\underline{Q}$), we could (with "equal reasonableness") define $C_{\pm 0}$^$\underline{C}_{-1}$ as:

$C_{\pm 0}$^$\underline{C}_{-1}$:= [($C_{\pm 0}$)$^{+1}$]$^{\wedge(-1)}$ = [($C_{\pm 0}$)]$^{(+1)\wedge(-1)}$ = ($C_{\pm 0}$)$^{(+1)}$ = $C_{\pm 0}$, & similarly: $C_{\pm 0}$^$\underline{C}_{+1}$:= ($C_{\pm 0}$)$^{(+1)}$ = $C_{\pm 0}$.

Thus, **Figure A3-1(a)** shows the *ordinary exponentiation* **^** in **Z**, while **Figure A3-1(b)** shows *the special exponentiation* '**^**' in *our symmetric* **Z'**: **W ∪ (–W)**.

Figure A3-1: " Exponentiation Tables for **{ –1, ±0, +1 }** in **Z** (left) and in 'Symmetric **Z**' " (right).

^	**–1**	**±0**	**+1**		'**^**'	**–1**	**±0**	**+1**
–1	**–1**	**+1**	**–1**		**–1**	**–1**	**–1**	**–1**
±0	undefined	Limit: **+1**	**±0**		**±0**	:= **±0**	Limits: **±1**	**±0**
+1	**+1**	**+1**	**+1**		**+1**	**+1**	**+1**	**+1**

Using results of the special '**^**' in 'symmetric **Z**', we filled in a "Possible Exponentiation Table for **{** $\underline{q}_{-1}$, $q_{\pm 0}$, $\underline{q}_{+1}$ **}**", shown as **Figure A3-2** below (assuming that **Cum ^** applies to **q**ualifier "**^**": $\underline{C}_1$:= $\underline{q}_{+1}$, C_0 := $q_{\pm 0}$, $\underline{C}_{-1}$:= $\underline{q}_{-1}$, and to the implied multiplication ×× and exponentiation "**^**" on $_{z}\underline{Q}$ elements). The interested reader may wish to attempt the research needed in order to extend this table beyond the set **{** $\underline{q}_{-1}$, $q_{\pm 0}$, $\underline{q}_{+1}$ **}** as base **&** exponent set.

Figure A3-2: "Possible Exponentiations Table for the values-set **{** $\underline{q}_{-1}$, $q_{\pm 0}$, $\underline{q}_{+1}$ **}**."

^ = "^"	$\underline{q}_{-1}$:= $\underline{C}_{-1}$	$q_{\pm 0}$:= $C_{\pm 0}$	$\underline{q}_1$:= $\underline{C}_{+1}$
$\underline{q}_{-1}$:= $\underline{C}_{-1}$	$\underline{q}_{-1}$:= $\underline{C}_{-1}$	$\underline{q}_{-1}$:= $\underline{C}_{-1}$	$\underline{q}_{-1}$:= $\underline{C}_{-1}$
$q_{\pm 0}$:= $C_{\pm 0}$	$q_{\pm 0}$:= $C_{\pm 0}$	$\underline{q}_{-1}$:= $\underline{C}_{-1}$/**/$\underline{q}_{+1}$:= $\underline{C}_{+1}$	q_0 := $C_{\pm 0}$
$\underline{q}_{+1}$:= $\underline{C}_{+1}$	$\underline{q}_{+1}$:= $\underline{C}_{+1}$	$\underline{q}_1$:= $\underline{C}_{+1}$	$\underline{q}_{+1}$:= $\underline{C}_{+1}$

-- *Joy-to-You!*